The Hyperbolic Plane

Experimental Algebra & Geometry Lab[†]

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Outline

What does the hyperbolic plane look like?

You can't see it all at once, its nature defies 3 dimensional space!

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A Quote

Many people have an impression, *based on years of schooling,* **that mathematics is** *an austere and formal subject concerned with* **complicated and ultimately confusing** *rules for the manipulation of numbers, symbols, and equations,* **rather like** *the preparation of* **a complicated income tax return**, *where there are myriad unexplained steps, rules, exceptions, and gotchas.* **Good mathematics is quite the opposite** *to this. Mathematics is an art of human understanding.*

-William Thurston

Forward of *Crocheting Adventures with Hyperbolic Planes*

Two more...

> **The mathematician's patterns, like the painter's or poet's, must be beautiful;** *the ideas, like the colours or the words, must fit together in a harmonious way.* **Beauty is the first test: there is no permanent place in the world for ugly mathematics.**

-G.H. Hardy
A Mathematician's Apology

> **When I could find voice, I shrieked aloud in agony, 'Either this is madness or it is Hell.'** *'It is neither,' calmly replied the voice of the Sphere,* **'it is Knowledge;** *it is Three Dimensions:* **open your eye once again and try to look steadily'**

-A. Square
Flatland

The story...

- A long time ago (3000 years or so), in a galaxy far far away (Greece), there was a geometer going by Euclid.
- He layed out a beautiful set of fundamental truths (axioms) about nature and proceded to use them to irrefutably prove all that could be said about geometry

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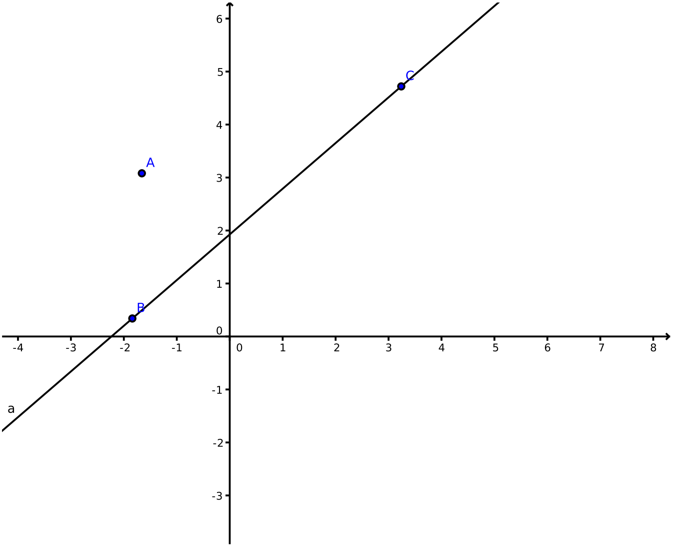
- He layed out a beautiful set of fundamental truths (axioms) about nature and proceded to use them to irrefutably prove all that could be said about geometry...or so he thought.
- Unfortunately, he was wrong.
- One of his axioms stood out like a sore thumb. It appeared that it should not be assumed, but proved like any other provable statement in mathematics (we call these theorems).
- It goes like this: for every line ℓ and every point P off that line, there is a unique line m parallel to ℓ that goes throught P.
- Try it with a ruler on a flat sheet of paper and you will see what I mean.

Euclidean Parallel Postulate



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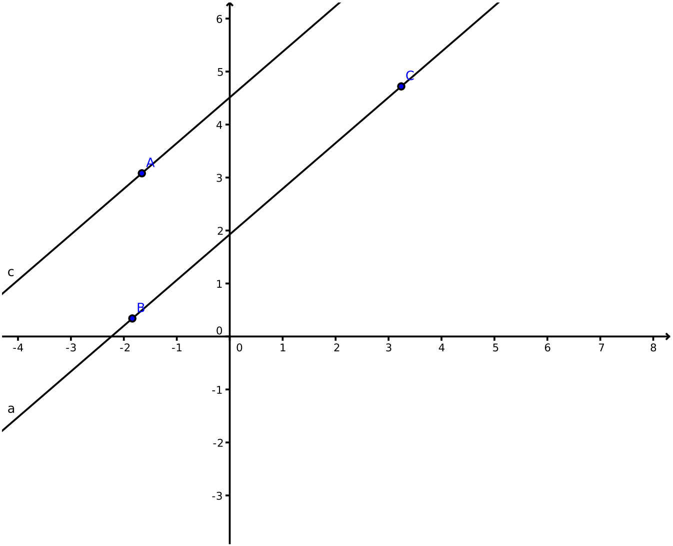

Spherical Lines

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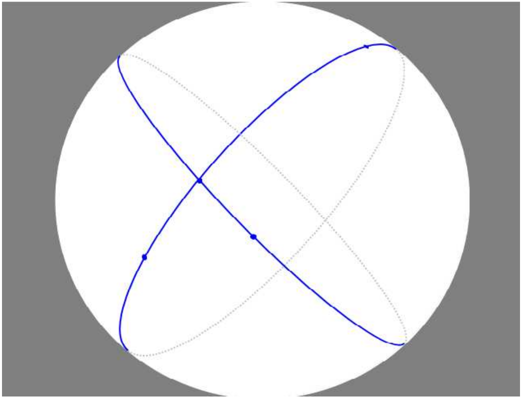
On a sphere, these are given by great circles, which cut the sphere in two equal pieces:

Elliptic Parallel Postulate



Therefore, parallel lines DO NOT EXIST on the sphere!

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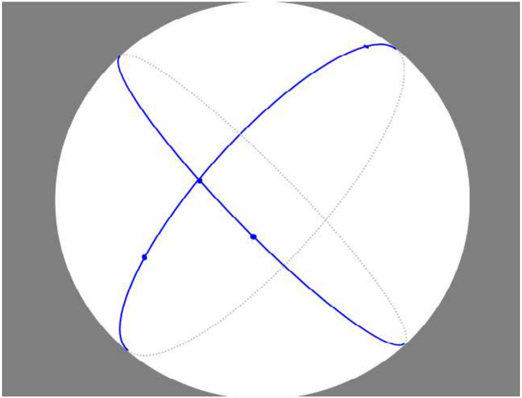

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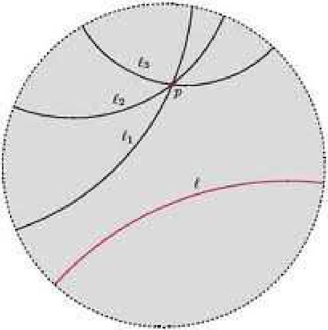
Since any two "half-cuts" must intersect!

1 or 0...perhaps more?



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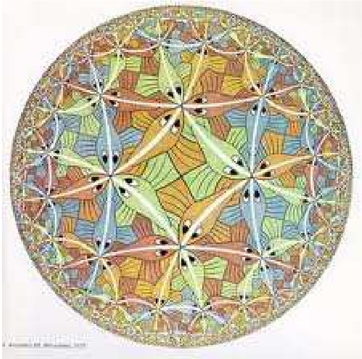
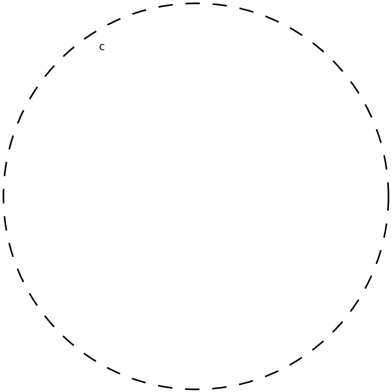


M.C. Escher

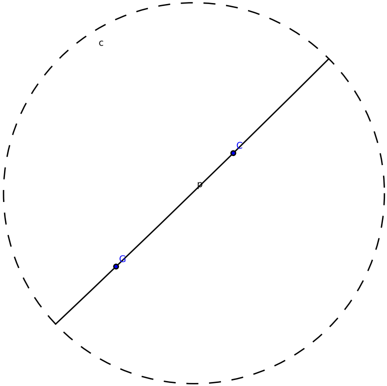
Points

Imagine the 2-D universe is the inside of a disk (if was big enough one would never know it wasn't a disk, would they?):



Lines

In this universe **distance** is measured so the shortest distance between two points is along a **diameter** :



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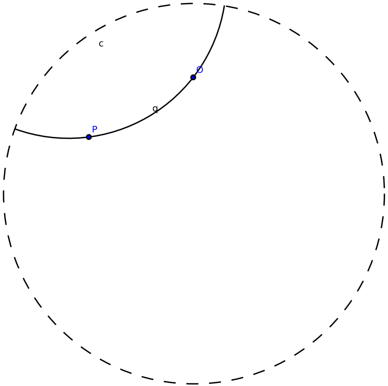

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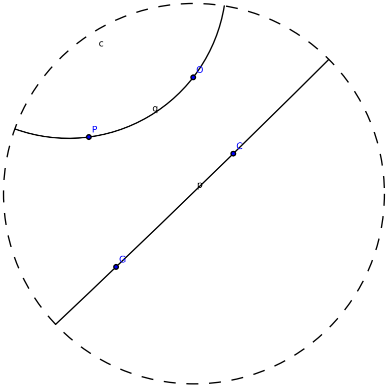
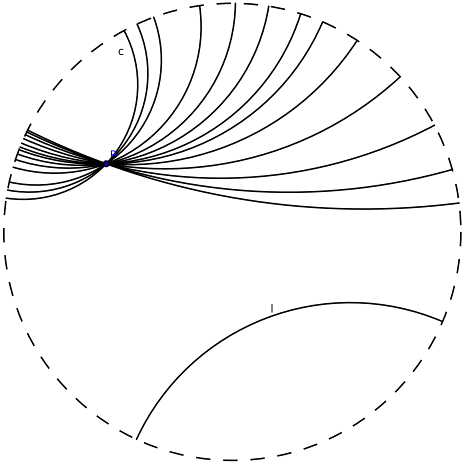


Hyperbolic Parallel Postulate

Then for every line ℓ and every point P off that line there exists infinitely many parellel lines through P to ℓ:



For 2000 years following Euclid geometers tried to prove that hyperbolic geometry could not exist. In the early 1800's Bolyai, Gauss, and Lobachevsky independently showed that hyperbolic geometry did exist.

Then in the late 1800's Beltrami, Klein, and Poincaré showed that one can construct explicit models of hyperbolic geometry. In particular, the disk universe we just discussed is one such model. To prove this one needs to define distance and angle measure in the disk so that lines do in fact appear as I described them.

In 1901 Hilbert showed that it was impossible for the hyperbolic plane to exist in 3-D (smoothly)! However, we can embed it smoothly in 6-D and we can smoothly construct "pieces" of the hyperbolic plane in 3-D.

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The Nash Embedding theorem (Nash from the movie *A Beautiful mind*) shows it is possible to embed it in a large Euclidean Space, Kuiper in 1955 showed an embedding was possible in 3D with curvature singularities.

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This model is possible since *unlike in Euclidean geometry*, 7 equalateral triangles can be put around a single vertex in the hyperbolic plane.

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Video of Creating Hyperbolic Plane

First Hyperbolic Geometry Theorem: The hyperbolic plane is FLOPPY!

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PROOF: Just look!

Some Neat Facts about Hyperbolic Geometry

1. The sum of the angle measures of a triangle in the hyperbolic plane is  LESS than 180 degrees.

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2. For every line ℓ and every point off that line P there are at least two  lines through P and parallel to ℓ.

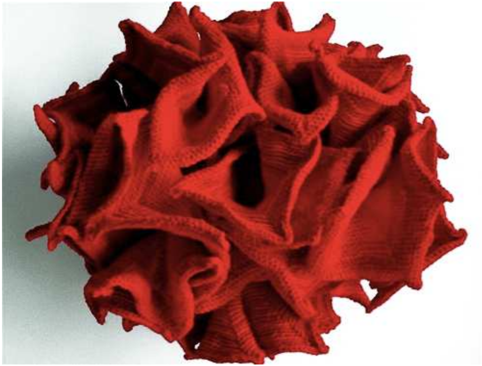
Reality?

1. The shape of the universe is unknown, and so it is possible that parts of its geometry are in fact hyperbolic. Hyperbolic geometry plays an important role is special relativity (in Physics) by unifying both Euclidean geometry of space and a model of hyperbolic space for time (why does the physics work like this?)
2. On the other hand, hyperbolic geometry occurs naturally in biology in coral reefs, sea slugs, and leaves and flowers (why does nature sometimes prefer hyperbolic geometry?)

In computer science hyperbolic geometry is import as a way to organize

and retrieve data.

In the arts, it is now in fashion (jewelry and clothing) and in abstract art pieces. We hope to contribute *your* work in an art installation for art walk here in the valley!



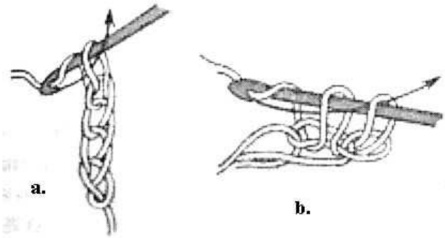
Daina Taimiņa (Cornell University) exhibits this piece at the Cooper-Hewitt National Design Museum

Margaret and Christine Wertheim (Institute For Figuring) exhibits this installation at the Smithonian

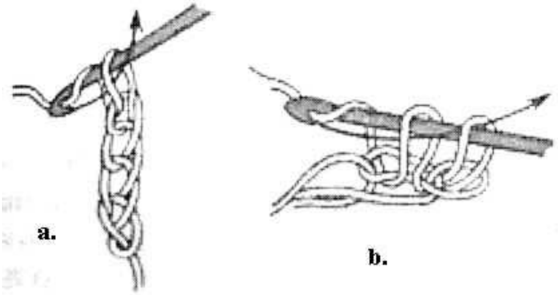
Instructions

Now you are ready to start the stitches:

1. Create a chain, 20 stitches for the beginning. See Figure (a).



2. For first stitch in each row insert hook into second chain from the hook. Take yarn over and pull through chain, leaving 2 loops on hook. Take yarn over and pull through both loops. **One single crochet stitch has been completed.** See Figure (b).



1. For the next N stitches proceed exactly like the first stitch except insert the hook into the next chain (instead of the 2nd).
2. For the $N+1$ stitch proceed as before except insert the hook into the same loop as the N-th stitch.
3. Repeat Steps 3 and 4 until you reach the end of the row.
4. At the end of the row before going to the next row do one extra chain stitch.

Acknowlegements

We were inspired by Daina Taimina and used her book "Crocheting Adventures with Hyperbolic Planes", published by AK Peters.

Most images are from various sources on the web (google images):

- Wikipedia:
  `http://en.wikipedia.org/wiki/Hyperbolic_geometry`
- Daina Taimiņa's websites:
  `http://www.math.cornell.edu/~dtaimina/`
- The Institute for Figureing: `http://theiff.org/`.

Some images created in GeoGebra: `http://www.geogebra.org/cms/`.

Thanks!

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- We thank Daina Taimiņa for giving us permission to use her ideas.