Shadows of a Flat Torus

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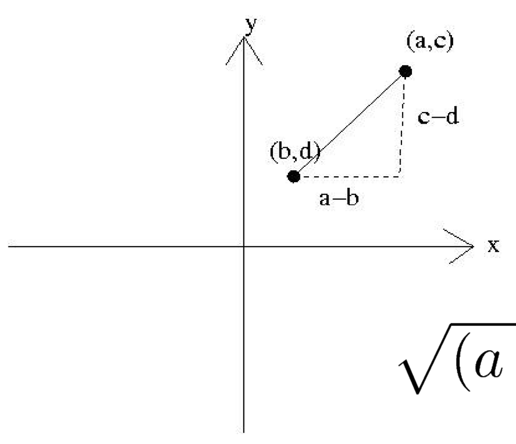
Introduction

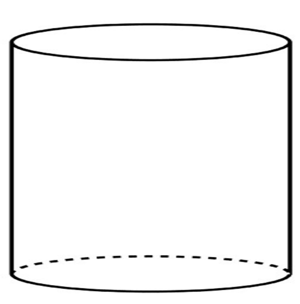
Imposing Euclidean geometry on a surface makes it everywhere flat, and surprisingly it is possible to do this on a torus!



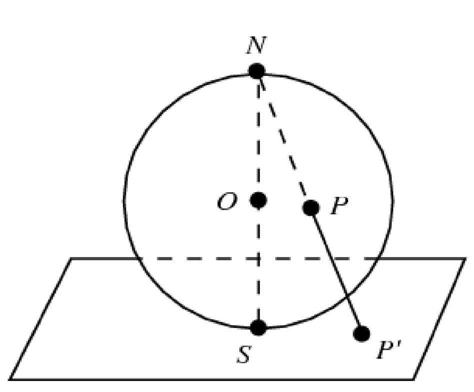
$$\sqrt{(a-b)^2+(c-d)^2}$$

Note that it is not possible to have a flat sphere (maps of the Earth are always distorted), and some tori are not everywhere flat (think of the round surface of a doughnut). The existence of a flat torus in 4D means that one can take a sheet of paper and tape its edges together in 4D without creating creases or corners (which is impossible in 3D).

For instance, consider the flat cylinder:



We determine the formulaic description for producing such a torus in 4D, and also for realizing its shadow in 3D by shining a light over it in 4D (a process called stereographic projection).



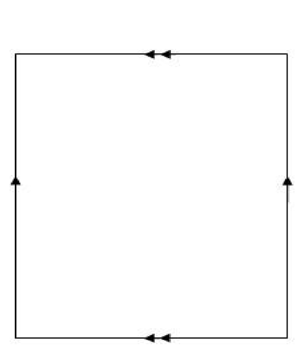
Additionally we verified that it was flat (computationally). We utilized the program *Mathematica* for these tasks, and also for all visualizations. We then computed and coded the orthogonal matrices necessary to rotate the image in 4D to visualize its shadow from all perspectives.

There are many different flat tori in 4D. Lastly, we created a continuous deformation of flat embedded tori which corresponds to a line of flat tori in the space of all flat tori.

This project was inspired by work of Thomas Banchoff at Brown University (`www.math.brown.edu/~banchoff`)

Methodology

The flat torus, denoted T^2, "unzipped" is given by:



After identifying opposite edges by translations, we obtain a flat torus parametrized by $S^1 x S^1$.

Methodology (continued)

We embed $S^1 x\, S^1$ in $R^2 x\, R^2 = R^4$ since $S^1 \subset R^2$ by: $(x,y) \mapsto \left(\frac{\cos(x)}{\sqrt{2}}, \frac{\sin(x)}{\sqrt{2}}, \frac{\cos(y)}{\sqrt{2}}, \frac{\sin(y)}{\sqrt{2}}\right)$

Note: $S^1 x S^1$ is homeomorphic to T^2 but they are not isometric.

The 3-sphere is denoted by $S^3 \subset R^4$. The flat torus, T^2, is in S^3. This is verified by

$$\left(\frac{\cos(x)}{\sqrt{2}}\right)^2+\left(\frac{\sin(x)}{\sqrt{2}}\right)^2+\left(\frac{\cos(y)}{\sqrt{2}}\right)^2+\left(\frac{\sin(y)}{\sqrt{2}}\right)^2=1$$

Since the identification by Euclidean distance preserving mappings (translations) corresponds to this embedding we conclude that the resulting space is flat (this can be independently verified by calculating the Gaussian curvature).

We then stereographic project the image from R^4 down to 3-space to produce its shadow by:

$$(x,y,z,w) \mapsto \left(\frac{x}{1-w}, \frac{y}{1-w}, \frac{z}{1-w}\right)$$

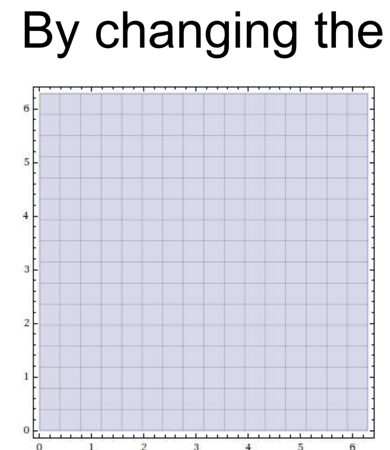
In order to get different shadows we then must apply the orthogonal matrices in R^4 to spin S^3 (there are 6 generators):

$$rotXY=\begin{bmatrix}\cos(\theta) & \sin(\theta) & 0 & 0\\ -\sin(\theta) & \cos(\theta) & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix} \quad rotYZ=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & \cos(\theta) & \sin(\theta) & 0\\ 0 & -\sin(\theta) & \cos(\theta) & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

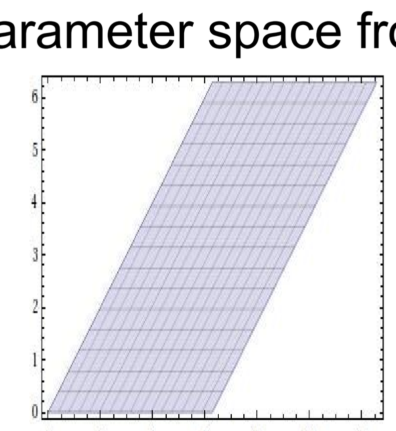
$$rotXZ=\begin{bmatrix}\cos(\theta) & 0 & -\sin(\theta) & 0\\ 0 & 1 & 0 & 0\\ \sin(\theta) & 0 & \cos(\theta) & 0\\ 0 & 0 & 0 & 1\end{bmatrix} \quad rotXU=\begin{bmatrix}\cos(\theta) & 0 & 0 & \sin(\theta)\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ -\sin(\theta) & 0 & 0 & \cos(\theta)\end{bmatrix}$$

$$rotYU=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & \cos(\theta) & 0 & -\sin(\theta)\\ 0 & 0 & 1 & 0\\ 0 & \sin(\theta) & 0 & \cos(\theta)\end{bmatrix} \quad rotZU=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & \cos(\theta) & -\sin(\theta)\\ 0 & 0 & \sin(\theta) & \cos(\theta)\end{bmatrix}$$

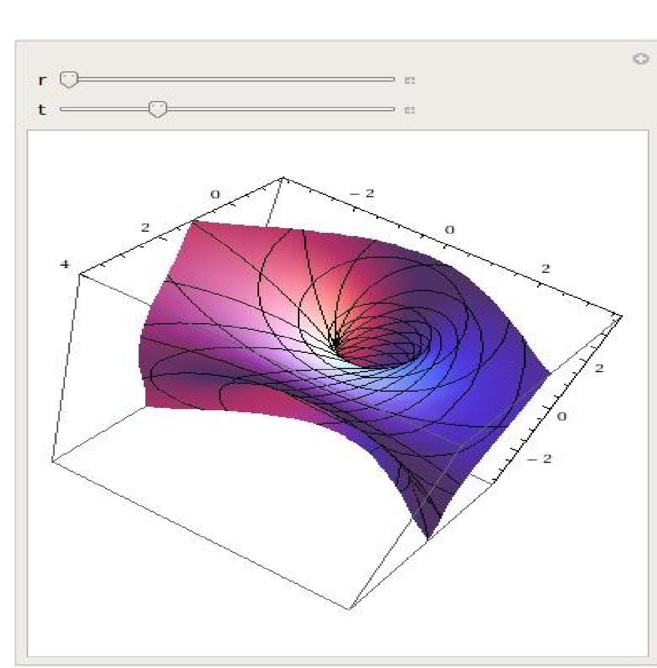
By changing the parameter space from



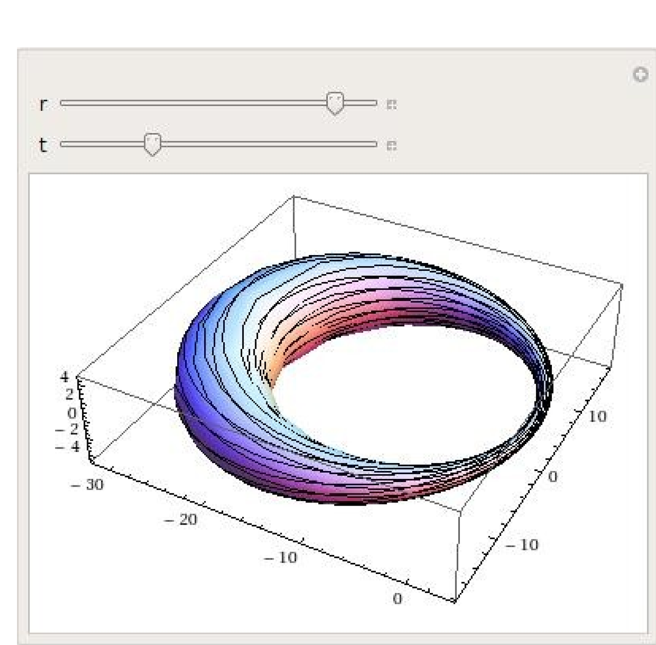
to



the shadow changes (at fixed perspective) from

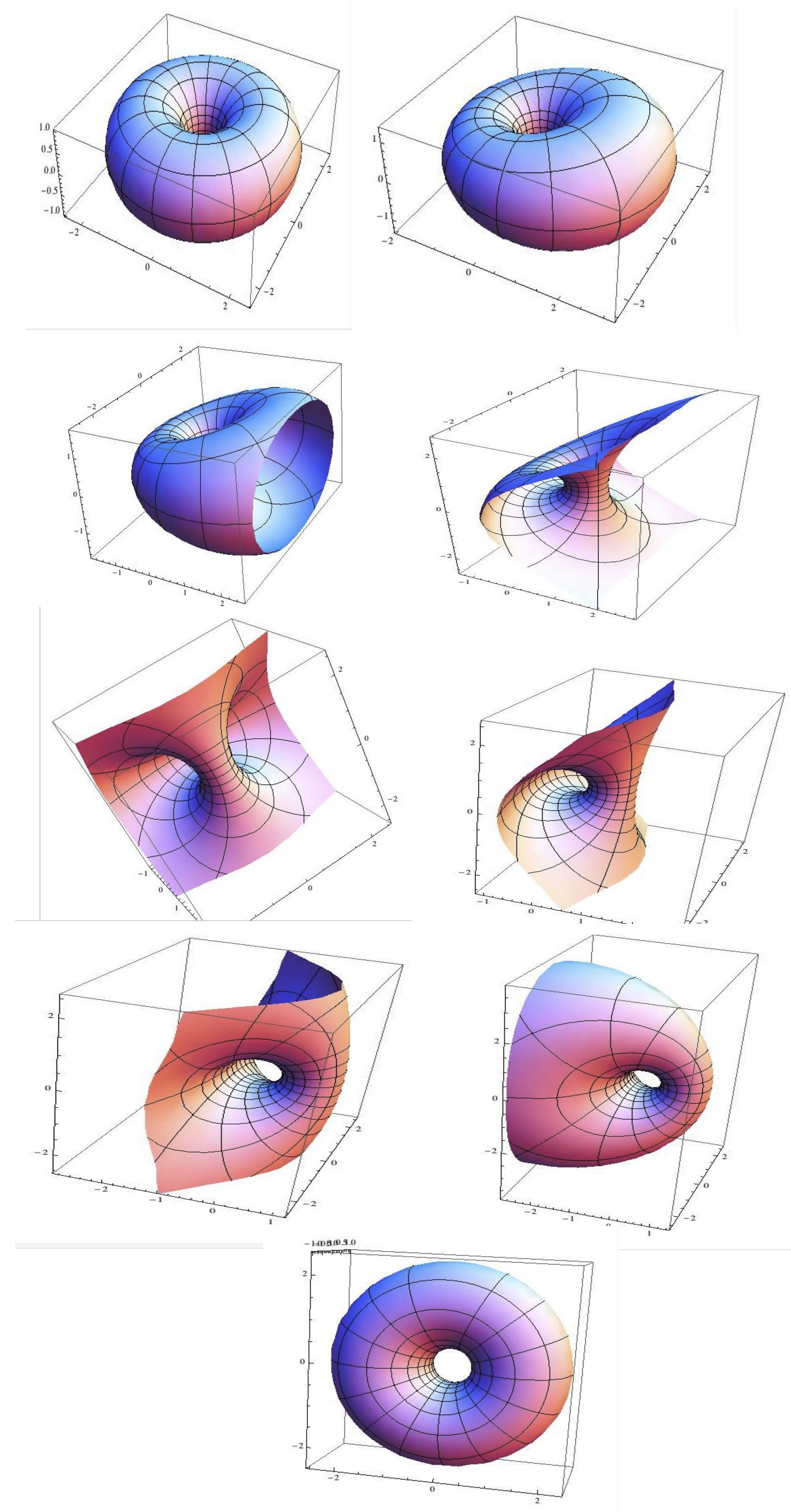


to



Results

Here follows the shadow of a full rotation of the flat torus in one rotational direction:



Summary

We demonstrated the existence of a flat torus and that it can only exist is 4D (or high dimensional space). After verifying that it is in fact a flat surface, we then verified that it is additionally in a 3-dimensional sphere which allowed for stereographic projection into familiar 3-space. We then visualized its shadow in 3D from all possible perspectives by continuously changing rotations via rotation matrices. Lastly, we deformed the flat structure itself and visualized how the shadow changes as the flat geometry varies.

Further studies

1. The space of *all* flat tori forms a plane. Having visualized lines of flat tori via flat embeddings, our next task is to visualize the entire space of flat tori via embeddings.

2. All other compact surfaces only admit Hyperbolic geometry. So after realizing the spaces of geometry on a sphere (only 1 spherical geometry exists!) and a torus via metric embeddings, we then would like to parametrize and visualize Hyperbolic surfaces. However this will be much more difficult since 4D is now too small and it is not clear how to explicitly describe the embeddings or the shadows.