Counting Representations of Free Groups

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Abstract

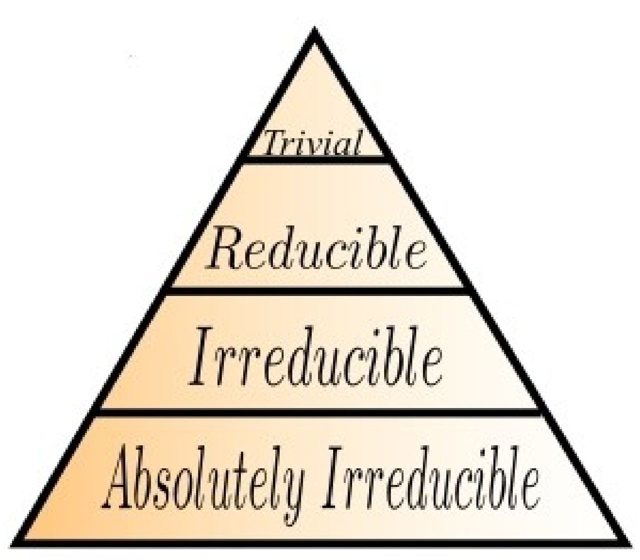
The collection of representations of a free group into a Lie group G corresponds to pointed bundles over a surface with boundary. As these are affine varieties, and thus field independent, we can ask how many points these spaces have over finite fields. We will answer this question for all free groups, and when G is $SL_n(\mathbb{Z}_p)$. We will also divide the set of representations into conjugation invariant strata, and count the number of representations in each strata.

Definitions

1. *Free Group*: A group with no relations between its group generators other than those imposed by the group axioms.
2. The *rank* r of a free group denotes the number of generators. For example, $F_3 =< x_1, x_2, x_3 >$ is a free group of rank 3.
3. $SL_n(\mathbb{Z}_p)$ denotes the *special linear group* over $\mathbb{Z}_p$, where p is a prime number. This set contains all determinant 1 matrices with entries in $\mathbb{Z}_p$
4. *Representation of a Free Group*: A homomorphism from F_r to $SL_n(\mathbb{Z}_p)$. The collection of all representations will be denoted by: $Hom(F_r, SL_n(\mathbb{Z}_p))$.

Methods

We wrote a program in *Mathematica* which allowed us to view $SL_n(\mathbb{Z}_p)$, for various n and p . We were then able to conjecture, and prove, a counting function for $SL_n(\mathbb{Z}_p)$. We knew that the total number of homomorphisms from F_r to any group G is just $|G|^r$, so we used this to count the homomorphisms when G is $SL_n(\mathbb{Z}_p)$. Once we had patterns we looked to archives of observed patterns to match our observations. Once we determined formulas, we proved each result we obtained. We then wrote programs to explore conjugation invariant strata in much the same way. We stratified the sets by stabilizer size.



Depiction of Conjugation Invariant Strata

Have a counting function for the total number of representations and a counting function for the first three strata gives us a means to compute the counting function for the last strata. We then can count conjugation orbits in each strata and thereby count the total number of orbits.

Findings

$$|SL_n(\mathbb{Z}_p)| = p^{\frac{n(n-1)}{2}} \prod_{i=2}^{n} (p^i - 1)$$

$$|Hom(F_r, SL_n(\mathbb{Z}_p)| = \left(p^{\frac{n(n-1)}{2}} \prod_{i=2}^{n} (p^i - 1)\right)^r$$

When $n = 2$, the set of representations divides into 5 sub-strata. It is possible to have an empty strata, which occurs to the second strata when $p = 2$ and $p = 3$.

Let λ be an integer such that $\lambda^n \equiv 1 \mod p$. The center of $SL_n(\mathbb{Z}_p)$ is composed of all matrices λI . The first strata consists of representations in the center. We are working on the next two strata with partial results.

Conclusion

We have found the counting polynomials for the representations of a free group of rank r to $SL_n(\mathbb{Z}_p)$. We have recently conjectured a counting formula for each strata (in the case *n*=2), and are currently working on a proof. Next we will generalize this to arbitrary values of *n*.