The Topology of Geodesic Chains

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Michael Fischer[1], Raul Mercado Jr.[1], and Dr. Sean Lawton

Department of Mathematics, The University of Texas-Pan American

[1]Louis Stokes Alliances for Minority Participation (LSAMP) Program

Introduction

Let S^2 be the 2-sphere; that is, the surface of a beach ball or the solutions to

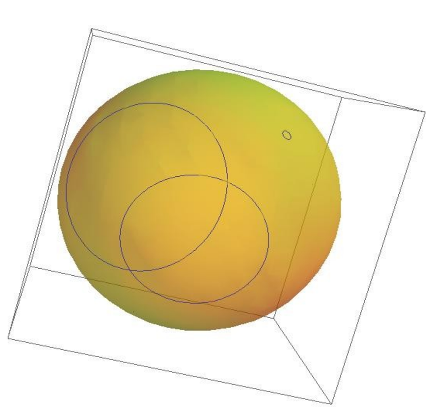
$$x^2+y^2+z^2=1$$

Light emanating from the north pole of S^2 defines a correspondence between the points of S^2 and the plane tangent to the south pole (an infinite flat sheet of paper touching only the south pole). This correspondence is called stereographic projection. Precisely, it is given by the formula

$$(x,y,z)\mapsto\left(\frac{x}{1-z},\frac{y}{1-z}\right)$$

Any path in S^2 that corresponds via stereographic projection to a Euclidean line or circle in the plane is called a *chain*.

A *topology* on a set is a collection of data which allows one to understand when two objects are close to each other. Formally, it is a collection of subsets, called *open sets*, that includes the entire space and the empty set, and is closed under unions and finite intersections. Since we can see when a circle or line appears close to another, the set of all chains *C* has the structure of a topological space.



There are exactly three constant curvature 2-dimensional geometries: *Euclidean, Spherical, and Hyperbolic*. Each of these can be realized as a sub-geometry of the angle-preserving geometry of chains on S^2. Thus, we can realize Euclidean lines, Spherical lines, and Hyperbolic lines as chains on S^2.

Let *EC* denote the set of all chains that are Euclidean lines, let *SC* denote the set of all chains that are Spherical lines, and let *HC* denote the set of all chains that are Hyperbolic lines. Each of these inherits a topology from *C*. We determine what these topologies are in familiar terms.

Methodology (summary)

1. Prove that chains are equivalent to real lines of negative determinant 2x2 Hermitian matrices over the complex numbers.
2. Describe subspaces of geodesic chains using geometric properties of Euclidean, Spherical, and Hyperbolic space.
3. Relate obtained spaces to well understood topological spaces to solve the problem.

Results

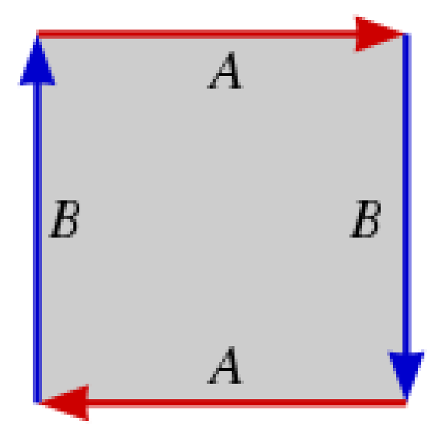
The topology of spherical chains *SC* is the space of lines in 3-dimensional space through the origin, known as:

$$\mathbb{RP}^2$$

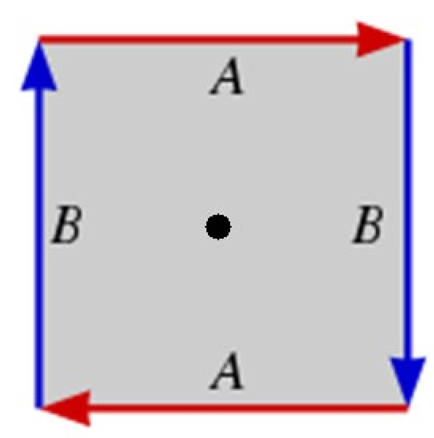
This space is equivalent to a Möbius band with a disc sewn to its edge.

In 3D there isn't enough space to actually sew a disc along the edge of the band because the disc would eventually have to cross through the band, so this has to be done in 4D.

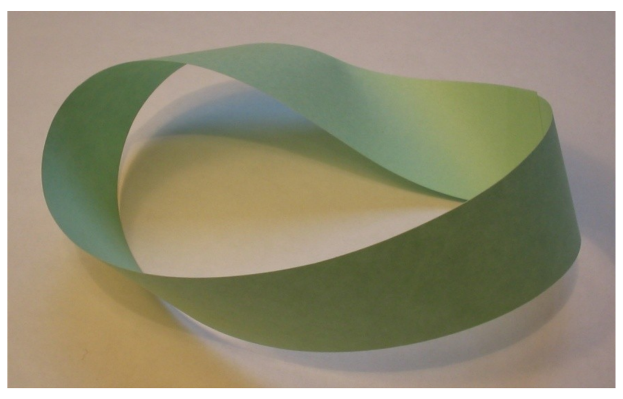
Here is what it looks like "unzipped":



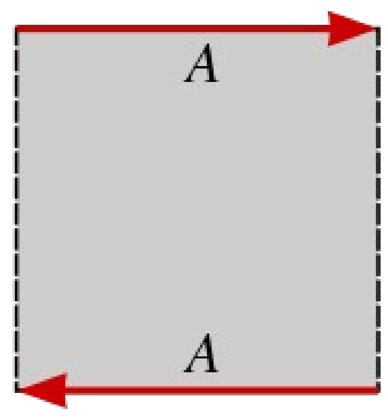
The topology on Euclidean chains *EC* is $\mathbb{RP}^2$ minus the zero vector.



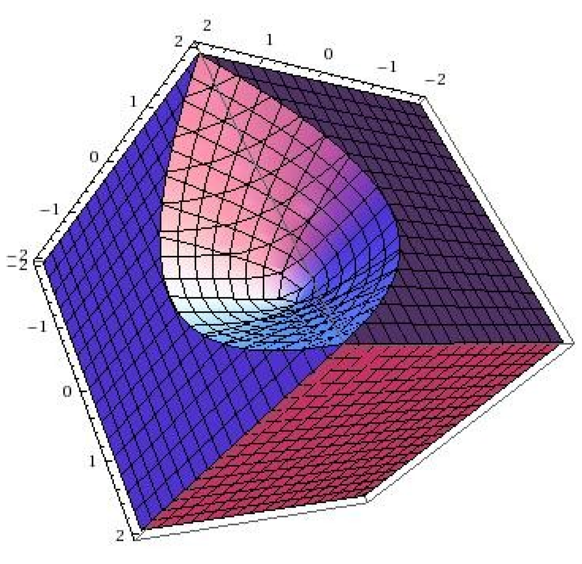
The resulting space is an open Möbius band (without an edge).



Here is the Möbius band "unzipped":



The topology of the space of Hyperbolic chains *HC* is a closed Möbius band (with an edge).



The above image is the region of lines through the origin removed from 3-space.

Methodology (technical)

$$\mathcal{C}\cong\mathcal{H}/\mathbb{R}^*,$$

Where

$$\mathcal{H}=\left\{\begin{pmatrix} a & b+ic \\ b-ic & d\end{pmatrix} \mid a,b,c,d\in\mathbb{R}, ad-b^2-c^2<0\right\}$$

To make the correspondence with lines or circles, one must take a Hermitian matrix from above and map it to

$$\begin{pmatrix} x+iy & 1\end{pmatrix}\begin{pmatrix} a & b+ic \\ b-ic & d\end{pmatrix}\begin{pmatrix} x-iy \\ 1\end{pmatrix}=0$$

The Euclidean chains are those chains that stereographically project to Euclidean lines in the plane. One shows that these are exactly those with *a=0*. Thus we conclude

$$\left\{\begin{pmatrix} 0 & b+ic \\ b-ic & d\end{pmatrix} \mid b,c,d\in\mathbb{R}, b^2+c^2>0\right\}/\mathbb{R}^*$$

is the space *EC*. This space is then equivalent to

$$\left(\mathbb{R}^3-\{(0,0,d)\mid d\in\mathbb{R}\}\right)/\mathbb{R}^*$$

which is the punctured real projective plane.

The other cases are handled similarly.

Summary

The (metric-less) study of the angle preserving (conformal) properties of chains on S^2 generalizes and unifies the study of Euclidean, Spherical, and Hyperbolic 2D geometry. This allows for the study of how different geometries, their objects and properties, relate to each other.

Some initial conclusions:

1. The space of all chains is the non-trivial line bundle over the real projective plane. (*this result was obtained in 2009 at UMCP with undergraduate Greg Laun*).
2. The space of Spherical chains is the real projective plane.
3. The space of Euclidean chains is an open Möbius band.
4. The space of Hyperbolic chains is a closed Möbius band.

Further Work

1. What are the chains that are both Hyperbolic and Euclidean?
2. What are the chains that are both Hyperbolic and Spherical?
3. What are the chains that are both Euclidean and Spherical?
4. What are the chains that are Euclidean, Spherical, and Hyperbolic?
5. What about the spaces of chain triangles in general and for each model geometry? How do they relate to each other?