Computing Groebner Bases for Toric Invariants of Generic Matrices.

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Cycle Monomials and Variables.

The set of all permutations, denoted S_n, is the set of all bijections of the set {1, 2, ..., n} for some positive integer n. S_n forms a group under composition. The below package adds in the command SymmetricGroup[n] which generates the ranges of all permuations as lists.

```
<< Combinatorica`
```

The following commands turn the set of permuations as a set of list into the set of permutations as actions.

```
PermutationList2PermutationAction[n_, k_] :=
  Module[{S = {}}, For[i = 1, i < n + 1, i++, S = Join[S, {i → SymmetricGroup[n][[k, i]]}]]; S];
SymmetricGroupActions[n_] := SymmetricGroupActions[n] =
   Table[PermutationList2PermutationAction[n, k], {k, 1, n!}];
```

```
SymmetricGroup[3]
SymmetricGroupActions[3]
```

```
{{1, 2, 3}, {1, 3, 2}, {2, 1, 3}, {2, 3, 1}, {3, 1, 2}, {3, 2, 1}}
```

```
{{1 → 1, 2 → 2, 3 → 3}, {1 → 1, 2 → 3, 3 → 2}, {1 → 2, 2 → 1, 3 → 3},
 {1 → 2, 2 → 3, 3 → 1}, {1 → 3, 2 → 1, 3 → 2}, {1 → 3, 2 → 2, 3 → 1}}
```

The conjugation action by diagonal invertible n×n matrices, which form a complex torus T_n on the set of all n×n matrices M_n determines an action on the complex polynomial ring in n^2 indeterminants $\{x_{i,j}$ | i,j are between 1 and n$\}$ denoted R_n. A cycle mononimial is a monomial defined as the product over the support of $\sigma \in S_n$ of all variables $x_{i,\sigma(i)}$. In symbols they are $\Pi_{i \in \mathrm{Supp}(\sigma)}\, x_{i,\sigma(i)}$. It can be shown that the monomials generate the ring of invariants of the above torus action ${R_n}^{T_n}$ which in turn in the coordinate ring of the algebraic quotient $M_n \;//\; T_n$.

The following commands define the matrix variables $x_{i,j}$ and the t_i variables.

```
Subscript[X, n_] := Table[Subscript[x, i, j], {i, 1, n}, {j, 1, n}];
MatrixVariables[n_] := Module[{S = {}},
   For[i = 1, i < n + 1, i++, For[j = 1, j < n + 1, j++, S = Union[S, {Subscript[X, n][[i, j]]}]]]; S];
TVariables[n_] := Module[{S = {}}, For[i = 1, i < n! - 1 + n + 1, i++, S = Join[S, {t_i}]]; S];
```

The following code takes the variables above and applies the symmetric group action to create the cycle monomials.

```
MatrixVariables[3]
TVariables[3]
```

$\{x_{1,1}, x_{1,2}, x_{1,3}, x_{2,1}, x_{2,2}, x_{2,3}, x_{3,1}, x_{3,2}, x_{3,3}\}$

$\{t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8\}$

```
p_ · x_{i_, j_} := x_{i, ({j} /. p)[[1]]};
Support[p_] :=
  Module[{S = {}}, For[i1 = 1, i1 < Length[p] + 1, i1++, If[(i1 /. p) ≠ i1, S = Join[S, {i1}]]]; S];
DiagonalInvariants[n_] := Module[{S = {}}, For[i2 = 1, i2 < n + 1, i2++, S = Join[S, {x_{i2, i2}}]]; S];
ListFactorsCycleMonomial[p_] := Module[{S = {}},
   For[i3 = 1, i3 < Length[Support[p]] + 1, i3++, S = Join[S, {p · x_{Support [p][[i3]], Support [p][[i3]]}}]]; S];
CycleMonomial[p_] := Apply[Times, ListFactorsCycleMonomial[p]];
CycleMonomials[n_] := CycleMonomials[n] = Module[{S = DiagonalInvariants[n]},
    For[i4 = 2, i4 < n! + 1, i4++, S = Union[S, {CycleMonomial[SymmetricGroupActions[n][[i4]]]}]]; S];
```

```
Support[{1 → 2, 2 → 1, 3 → 3}]
```

{1, 2}

```
DiagonalInvariants[3]
```

$\{x_{1,1}, x_{2,2}, x_{3,3}\}$

```
CycleMonomials[5]
```

$$\{x_{1,1}, x_{1,2} x_{2,1}, x_{2,2}, x_{1,3} x_{3,1}, x_{1,2} x_{2,3} x_{3,1}, x_{1,3} x_{2,1} x_{3,2}, x_{2,3} x_{3,2}, x_{3,3}, x_{1,4} x_{4,1}, x_{1,2} x_{2,4} x_{4,1},$$
$$x_{1,4} x_{2,3} x_{3,2} x_{4,1}, x_{1,3} x_{2,4} x_{3,2} x_{4,1}, x_{1,3} x_{3,4} x_{4,1}, x_{1,2} x_{2,3} x_{3,4} x_{4,1}, x_{1,4} x_{2,1} x_{4,2}, x_{2,4} x_{4,2},$$
$$x_{1,4} x_{2,3} x_{3,1} x_{4,2}, x_{1,3} x_{2,4} x_{3,1} x_{4,2}, x_{1,3} x_{2,1} x_{3,4} x_{4,2}, x_{2,3} x_{3,4} x_{4,2}, x_{1,4} x_{3,1} x_{4,3}, x_{1,2} x_{2,4} x_{3,1} x_{4,3},$$
$$x_{1,4} x_{2,1} x_{3,2} x_{4,3}, x_{2,4} x_{3,2} x_{4,3}, x_{3,4} x_{4,3}, x_{1,2} x_{2,1} x_{3,4} x_{4,3}, x_{4,4}, x_{1,5} x_{5,1}, x_{1,2} x_{2,5} x_{5,1},$$
$$x_{1,5} x_{2,3} x_{3,2} x_{5,1}, x_{1,3} x_{2,5} x_{3,2} x_{5,1}, x_{1,3} x_{3,5} x_{5,1}, x_{1,2} x_{2,3} x_{3,5} x_{5,1}, x_{1,5} x_{2,4} x_{4,2} x_{5,1}, x_{1,4} x_{2,5} x_{4,2} x_{5,1},$$
$$x_{1,5} x_{2,3} x_{3,4} x_{4,2} x_{5,1}, x_{1,3} x_{2,5} x_{3,4} x_{4,2} x_{5,1}, x_{1,4} x_{2,3} x_{3,5} x_{4,2} x_{5,1}, x_{1,3} x_{2,4} x_{3,5} x_{4,2} x_{5,1},$$
$$x_{1,5} x_{2,4} x_{3,2} x_{4,3} x_{5,1}, x_{1,4} x_{2,5} x_{3,2} x_{4,3} x_{5,1}, x_{1,5} x_{3,4} x_{4,3} x_{5,1}, x_{1,2} x_{2,5} x_{3,4} x_{4,3} x_{5,1}, x_{1,4} x_{3,5} x_{4,3} x_{5,1},$$
$$x_{1,2} x_{2,4} x_{3,5} x_{4,3} x_{5,1}, x_{1,4} x_{4,5} x_{5,1}, x_{1,2} x_{2,4} x_{4,5} x_{5,1}, x_{1,4} x_{2,3} x_{3,2} x_{4,5} x_{5,1}, x_{1,3} x_{2,4} x_{3,2} x_{4,5} x_{5,1},$$
$$x_{1,3} x_{3,4} x_{4,5} x_{5,1}, x_{1,2} x_{2,3} x_{3,4} x_{4,5} x_{5,1}, x_{1,5} x_{2,1} x_{5,2}, x_{2,5} x_{5,2}, x_{1,5} x_{2,3} x_{3,1} x_{5,2}, x_{1,3} x_{2,5} x_{3,1} x_{5,2},$$
$$x_{1,3} x_{2,1} x_{3,5} x_{5,2}, x_{2,3} x_{3,5} x_{5,2}, x_{1,5} x_{2,4} x_{4,1} x_{5,2}, x_{1,4} x_{2,5} x_{4,1} x_{5,2}, x_{1,5} x_{2,3} x_{3,4} x_{4,1} x_{5,2},$$
$$x_{1,3} x_{2,5} x_{3,4} x_{4,1} x_{5,2}, x_{1,4} x_{2,3} x_{3,5} x_{4,1} x_{5,2}, x_{1,3} x_{2,4} x_{3,5} x_{4,1} x_{5,2}, x_{1,5} x_{2,4} x_{3,1} x_{4,3} x_{5,2},$$
$$x_{1,4} x_{2,5} x_{3,1} x_{4,3} x_{5,2}, x_{1,5} x_{2,1} x_{3,4} x_{4,3} x_{5,2}, x_{2,5} x_{3,4} x_{4,3} x_{5,2}, x_{1,4} x_{2,1} x_{3,5} x_{4,3} x_{5,2},$$
$$x_{2,4} x_{3,5} x_{4,3} x_{5,2}, x_{1,4} x_{2,1} x_{4,5} x_{5,2}, x_{2,4} x_{4,5} x_{5,2}, x_{1,4} x_{2,3} x_{3,1} x_{4,5} x_{5,2}, x_{1,3} x_{2,4} x_{3,1} x_{4,5} x_{5,2},$$
$$x_{1,3} x_{2,1} x_{3,4} x_{4,5} x_{5,2}, x_{2,3} x_{3,4} x_{4,5} x_{5,2}, x_{1,5} x_{3,1} x_{5,3}, x_{1,2} x_{2,5} x_{3,1} x_{5,3}, x_{1,5} x_{2,1} x_{3,2} x_{5,3},$$
$$x_{2,5} x_{3,2} x_{5,3}, x_{3,5} x_{5,3}, x_{1,2} x_{2,1} x_{3,5} x_{5,3}, x_{1,5} x_{2,4} x_{3,2} x_{4,1} x_{5,3}, x_{1,4} x_{2,5} x_{3,2} x_{4,1} x_{5,3},$$
$$x_{1,5} x_{3,4} x_{4,1} x_{5,3}, x_{1,2} x_{2,5} x_{3,4} x_{4,1} x_{5,3}, x_{1,4} x_{3,5} x_{4,1} x_{5,3}, x_{1,2} x_{2,4} x_{3,5} x_{4,1} x_{5,3}, x_{1,5} x_{2,4} x_{3,1} x_{4,2} x_{5,3},$$
$$x_{1,4} x_{2,5} x_{3,1} x_{4,2} x_{5,3}, x_{1,5} x_{2,1} x_{3,4} x_{4,2} x_{5,3}, x_{2,5} x_{3,4} x_{4,2} x_{5,3}, x_{1,4} x_{2,1} x_{3,5} x_{4,2} x_{5,3}, x_{2,4} x_{3,5} x_{4,2} x_{5,3},$$
$$x_{1,4} x_{3,1} x_{4,5} x_{5,3}, x_{1,2} x_{2,4} x_{3,1} x_{4,5} x_{5,3}, x_{1,4} x_{2,1} x_{3,2} x_{4,5} x_{5,3}, x_{2,4} x_{3,2} x_{4,5} x_{5,3}, x_{3,4} x_{4,5} x_{5,3},$$
$$x_{1,2} x_{2,1} x_{3,4} x_{4,5} x_{5,3}, x_{1,5} x_{4,1} x_{5,4}, x_{1,2} x_{2,5} x_{4,1} x_{5,4}, x_{1,5} x_{2,3} x_{3,2} x_{4,1} x_{5,4}, x_{1,3} x_{2,5} x_{3,2} x_{4,1} x_{5,4},$$
$$x_{1,3} x_{3,5} x_{4,1} x_{5,4}, x_{1,2} x_{2,3} x_{3,5} x_{4,1} x_{5,4}, x_{1,5} x_{2,1} x_{4,2} x_{5,4}, x_{2,5} x_{4,2} x_{5,4}, x_{1,5} x_{2,3} x_{3,1} x_{4,2} x_{5,4},$$
$$x_{1,3} x_{2,5} x_{3,1} x_{4,2} x_{5,4}, x_{1,3} x_{2,1} x_{3,5} x_{4,2} x_{5,4}, x_{2,3} x_{3,5} x_{4,2} x_{5,4}, x_{1,5} x_{3,1} x_{4,3} x_{5,4}, x_{1,2} x_{2,5} x_{3,1} x_{4,3} x_{5,4},$$
$$x_{1,5} x_{2,1} x_{3,2} x_{4,3} x_{5,4}, x_{2,5} x_{3,2} x_{4,3} x_{5,4}, x_{3,5} x_{4,3} x_{5,4}, x_{1,2} x_{2,1} x_{3,5} x_{4,3} x_{5,4}, x_{4,5} x_{5,4}, x_{1,2} x_{2,1} x_{4,5} x_{5,4},$$
$$x_{1,3} x_{3,1} x_{4,5} x_{5,4}, x_{1,2} x_{2,3} x_{3,1} x_{4,5} x_{5,4}, x_{1,3} x_{2,1} x_{3,2} x_{4,5} x_{5,4}, x_{2,3} x_{3,2} x_{4,5} x_{5,4}, x_{5,5}\}$$

The following code generates a set of differences between the cycle monomials (generating set for ring of invariants) and a set of free variables. Then a Groebner Basis Algorithm is used, with elimination, to determine the defining ideal of relations for the invariant ring.

```
DifferenceFunctions[n_] := Module[{S = {}, T = TVariables[n], C = CycleMonomials[n]},
   For[i5 = 1, i5 < n! - 1 + n + 1, i5++, S = Join[S, {T[[i5]] - C[[i5]]}]]; S];
InvariantBasis[n_] := GroebnerBasis[DifferenceFunctions[n],
   Union[MatrixVariables[n], TVariables[n]], MatrixVariables[n]];
```

```
DifferenceFunctions[3]
```

$$\{t_1 - x_{1,1}, t_2 - x_{1,2} x_{2,1}, t_3 - x_{2,2}, t_4 - x_{1,3} x_{3,1}, t_5 - x_{1,2} x_{2,3} x_{3,1}, t_6 - x_{1,3} x_{2,1} x_{3,2}, t_7 - x_{2,3} x_{3,2}, t_8 - x_{3,3}\}$$

Here we use this function to reproduce known examples and attempt to determine the 5×5 case, which is unknown as far as we know.

```
InvariantBasis[1]
```

```
{}
```

```
InvariantBasis[2]
```

```
{}
```

Therefore the 1×1 case and the 2×2 cases are affine space; that is, no relations.

```
InvariantBasis[3]
```

$\{-t_5\, t_6 + t_2\, t_4\, t_7\}$

Therefore, the 3×3 case is a hypersurface in 8 dimensional affine space with 3 free variables. This means it is all of affine 3 space crossed with a hyper surface in affine 5-space.

InvariantBasis[**4**]

$\{-t_{22} t_{23} t_{25} + t_{21} t_{24} t_{26}, -t_{19} t_{22} + t_{18} t_{26}, -t_{19} t_{21} t_{24} + t_{18} t_{23} t_{25}, -t_{20} t_{21} + t_{17} t_{25}, -t_{20} t_{22} t_{23} + t_{17} t_{24} t_{26},$
$-t_{18} t_{20} t_{23} + t_{17} t_{19} t_{24}, -t_{16} t_{23} + t_{15} t_{24}, t_{15} t_{22} t_{25} - t_{16} t_{21} t_{26}, t_{15} t_{20} t_{22} - t_{16} t_{17} t_{26},$
$-t_{16} t_{19} t_{21} + t_{15} t_{18} t_{25}, -t_{16} t_{17} t_{19} + t_{15} t_{18} t_{20}, -t_{14} t_{19} t_{25} + t_{13} t_{20} t_{26}, t_{13} t_{20} t_{22} - t_{14} t_{18} t_{25},$
$-t_{14} t_{19} t_{21} + t_{13} t_{17} t_{26}, -t_{14} t_{18} t_{21} + t_{13} t_{17} t_{22}, -t_{13} t_{24} + t_{12} t_{25}, -t_{13} t_{22} t_{23} + t_{12} t_{21} t_{26},$
$-t_{14} t_{19} t_{24} + t_{12} t_{20} t_{26}, t_{12} t_{20} t_{22} - t_{14} t_{18} t_{24}, t_{12} t_{20} t_{21} - t_{13} t_{17} t_{24}, t_{12} t_{19} t_{21} - t_{13} t_{18} t_{23},$
$-t_{14} t_{18} t_{23} + t_{12} t_{17} t_{26}, t_{12} t_{17} t_{19} t_{22} - t_{14} t_{18}^2 t_{23}, -t_{14} t_{23} + t_{11} t_{26}, -t_{14} t_{21} t_{24} + t_{11} t_{22} t_{25},$
$t_{11} t_{20} t_{22} - t_{14} t_{17} t_{24}, -t_{13} t_{20} t_{23} + t_{11} t_{19} t_{25}, -t_{12} t_{20} t_{23} + t_{11} t_{19} t_{24}, t_{11} t_{19} t_{22} - t_{14} t_{18} t_{23},$
$t_{11} t_{19} t_{21} - t_{13} t_{17} t_{23}, -t_{12} t_{17} + t_{11} t_{18}, t_{11} t_{16} t_{19} - t_{12} t_{15} t_{20}, -t_{12} t_{14} t_{21} + t_{11} t_{13} t_{22}, -t_{14} t_{16} + t_{10} t_{20},$
$t_{10} t_{19} t_{25} - t_{13} t_{16} t_{26}, t_{10} t_{19} t_{24} - t_{12} t_{16} t_{26}, t_{10} t_{19} t_{23} - t_{12} t_{15} t_{26}, t_{10} t_{19} t_{21} - t_{13} t_{15} t_{22},$
$-t_{13} t_{16} t_{22} + t_{10} t_{18} t_{25}, -t_{12} t_{16} t_{22} + t_{10} t_{18} t_{24}, -t_{12} t_{15} t_{22} + t_{10} t_{18} t_{23}, -t_{14} t_{15} t_{22} + t_{10} t_{17} t_{26},$
$-t_{11} t_{16} t_{22} + t_{10} t_{17} t_{24}, -t_{11} t_{15} t_{22} + t_{10} t_{17} t_{23}, -t_{14} t_{15} t_{18} + t_{10} t_{17} t_{19}, -t_{12} t_{14} t_{15} + t_{10} t_{11} t_{19},$
$-t_{10} t_{23} t_{25} + t_9 t_{24} t_{26}, -t_{10} t_{21} + t_9 t_{22}, -t_{14} t_{15} t_{25} + t_9 t_{20} t_{26}, t_9 t_{20} t_{24} - t_{11} t_{16} t_{25},$
$t_9 t_{20} t_{23} - t_{11} t_{15} t_{25}, -t_{13} t_{15} + t_9 t_{19}, -t_{13} t_{16} t_{21} + t_9 t_{18} t_{25}, -t_{12} t_{16} t_{21} + t_9 t_{18} t_{24},$
$-t_{12} t_{15} t_{21} + t_9 t_{18} t_{23}, -t_{13} t_{16} t_{17} + t_9 t_{18} t_{20}, -t_{14} t_{15} t_{21} + t_9 t_{17} t_{26}, -t_{11} t_{16} t_{21} + t_9 t_{17} t_{24},$
$-t_{11} t_{15} t_{21} + t_9 t_{17} t_{23}, -t_{10} t_{15} t_{25} + t_9 t_{16} t_{26}, t_9 t_{14} t_{24} - t_{10} t_{11} t_{25}, -t_{10} t_{13} t_{17} + t_9 t_{14} t_{18},$
$t_9 t_{14} t_{16} t_{23} - t_{10} t_{11} t_{15} t_{25}, -t_{10} t_{13} t_{23} + t_9 t_{12} t_{26}, -t_{11} t_{13} t_{16} + t_9 t_{12} t_{20}, -t_{10} t_{11} t_{13} + t_9 t_{12} t_{14},$
$-t_{20} t_{24} + t_7 t_{16} t_{25}, t_7 t_{16} t_{21} - t_{17} t_{24}, -t_{20} t_{23} + t_7 t_{15} t_{25}, t_7 t_{15} t_{21} - t_{17} t_{23}, t_7 t_{13} t_{16} - t_{12} t_{20},$
$t_7 t_{13} t_{15} - t_{11} t_{19}, -t_{14} t_{24} + t_7 t_{10} t_{25}, t_7 t_{10} t_{21} - t_{11} t_{22}, t_7 t_{10} t_{13} - t_{12} t_{14}, t_7 t_9 - t_{11}, -t_7 t_{19} + t_6 t_{20},$
$-t_7 t_{18} t_{23} + t_6 t_{17} t_{24}, -t_{19} t_{24} + t_6 t_{16} t_{25}, t_6 t_{16} t_{21} - t_{18} t_{23}, t_6 t_{16} t_{17} - t_7 t_{15} t_{18}, -t_{19} t_{23} + t_6 t_{15} t_{25},$
$t_6 t_{14} t_{25} - t_7 t_{13} t_{26}, t_6 t_{14} t_{24} - t_7 t_{12} t_{26}, t_6 t_{14} t_{16} - t_7 t_{10} t_{19}, t_6 t_{13} t_{16} - t_{12} t_{19}, -t_7 t_{13} t_{23} + t_6 t_{11} t_{25},$
$-t_7 t_{12} t_{23} + t_6 t_{11} t_{24}, -t_7 t_{12} t_{15} + t_6 t_{11} t_{16}, t_6 t_{10} t_{25} - t_{12} t_{26}, -t_{12} t_{22} t_{23} + t_6 t_{10} t_{21} t_{24},$
$t_6 t_{10} t_{13} t_{24} - t_{12}^2 t_{26}, -t_{13} t_{23} + t_6 t_9 t_{25}, -t_{12} t_{23} + t_6 t_9 t_{24}, -t_{12} t_{15} + t_6 t_9 t_{16}, -t_7 t_{22} + t_5 t_{24},$
$t_5 t_{23} t_{25} - t_7 t_{21} t_{26}, t_5 t_{20} t_{23} - t_7 t_{17} t_{26}, t_5 t_{19} t_{23} - t_6 t_{17} t_{26}, -t_6 t_{17} t_{22} + t_5 t_{18} t_{23}, -t_{20} t_{22} + t_5 t_{16} t_{25},$
$-t_7 t_{15} t_{22} + t_5 t_{16} t_{23}, t_5 t_{16} t_{21} - t_{17} t_{22}, t_5 t_{15} t_{25} - t_{17} t_{26}, t_5 t_{15} t_{20} t_{21} - t_{17}^2 t_{26}, -t_6 t_{14} t_{21} + t_5 t_{13} t_{23},$
$t_5 t_{13} t_{16} - t_{14} t_{18}, -t_{14} t_{17} t_{19} + t_5 t_{13} t_{15} t_{20}, -t_6 t_{14} t_{22} + t_5 t_{12} t_{26}, -t_6 t_{11} t_{22} + t_5 t_{12} t_{23},$
$-t_7 t_{14} t_{18} + t_5 t_{12} t_{20}, -t_6 t_{14} t_{18} + t_5 t_{12} t_{19}, t_5 t_{12} t_{16} - t_7 t_{10} t_{18}, t_5 t_{12} t_{15} - t_6 t_{10} t_{17},$
$-t_7 t_{14} t_{21} + t_5 t_{11} t_{25}, -t_7 t_{14} t_{17} + t_5 t_{11} t_{20}, -t_6 t_{14} t_{17} + t_5 t_{11} t_{19}, t_5 t_{11} t_{16} - t_7 t_{10} t_{17},$
$t_5 t_{11} t_{13} t_{15} - t_6 t_9 t_{14} t_{17}, -t_{14} t_{22} + t_5 t_{10} t_{25}, -t_{14} t_{21} + t_5 t_9 t_{25}, -t_{14} t_{17} + t_5 t_9 t_{20}, t_5 t_9 t_{16} - t_{10} t_{17},$
$-t_6 t_{22} t_{25} + t_4 t_{24} t_{26}, -t_6 t_{21} + t_4 t_{23}, -t_5 t_{19} t_{25} + t_4 t_{20} t_{26}, t_4 t_{20} t_{24} - t_7 t_{18} t_{25}, t_4 t_{20} t_{22} - t_5 t_{18} t_{25},$
$t_4 t_{19} t_{24} - t_6 t_{18} t_{25}, -t_5 t_{19} t_{21} + t_4 t_{17} t_{26}, -t_7 t_{18} t_{21} + t_4 t_{17} t_{24}, -t_5 t_{18} t_{21} + t_4 t_{17} t_{22}, t_4 t_{16} - t_{18},$
$-t_{19} t_{21} + t_4 t_{15} t_{25}, -t_{17} t_{19} + t_4 t_{15} t_{20}, -t_5 t_{13} + t_4 t_{14}, -t_6 t_{13} t_{22} + t_4 t_{12} t_{26}, -t_7 t_{13} t_{18} + t_4 t_{12} t_{20},$
$-t_6 t_{13} t_{18} + t_4 t_{12} t_{19}, t_4 t_{12} t_{15} - t_6 t_9 t_{18}, -t_7 t_{13} t_{21} + t_4 t_{11} t_{25}, -t_7 t_{12} t_{21} + t_4 t_{11} t_{24},$
$-t_5 t_{12} t_{21} + t_4 t_{11} t_{22}, -t_7 t_{13} t_{17} + t_4 t_{11} t_{20}, -t_6 t_{13} t_{17} + t_4 t_{11} t_{19}, t_4 t_{11} t_{15} - t_6 t_9 t_{17},$
$-t_{13} t_{22} + t_4 t_{10} t_{25}, -t_{12} t_{22} + t_4 t_{10} t_{24}, t_4 t_{10} t_{17} - t_5 t_9 t_{18}, t_4 t_{10} t_{11} - t_5 t_9 t_{12}, -t_{13} t_{21} + t_4 t_9 t_{25},$
$-t_{12} t_{21} + t_4 t_9 t_{24}, -t_{13} t_{17} + t_4 t_9 t_{20}, -t_5 t_6 t_{25} + t_4 t_7 t_{26}, t_4 t_7 t_{19} t_{22} - t_5 t_6 t_{18} t_{25}, t_4 t_7 t_{15} - t_6 t_{17},$
$t_4 t_7 t_{10} - t_5 t_{12}, t_2 t_{25} - t_{26}, -t_{22} t_{23} + t_2 t_{21} t_{24}, t_2 t_{20} t_{24} - t_7 t_{16} t_{26}, t_2 t_{20} t_{23} - t_7 t_{15} t_{26},$
$t_2 t_{20} t_{22} - t_5 t_{16} t_{26}, t_2 t_{20} t_{21} - t_{17} t_{26}, t_2 t_{19} t_{24} - t_6 t_{16} t_{26}, t_2 t_{19} t_{23} - t_6 t_{15} t_{26}, t_2 t_{19} t_{21} - t_4 t_{15} t_{26},$
$-t_6 t_{16} t_{22} + t_2 t_{18} t_{24}, -t_6 t_{15} t_{22} + t_2 t_{18} t_{23}, t_2 t_{18} t_{21} - t_4 t_{15} t_{22}, -t_5 t_{16} t_{19} + t_2 t_{18} t_{20},$
$-t_5 t_{15} + t_2 t_{17}, t_2 t_{16} t_{21} - t_{15} t_{22}, t_2 t_{14} t_{24} - t_7 t_{10} t_{26}, t_2 t_{14} t_{22} - t_5 t_{10} t_{26}, t_2 t_{14} t_{21} - t_5 t_9 t_{26},$
$t_2 t_{14} t_{18} - t_5 t_{10} t_{19}, t_2 t_{14} t_{16} t_{23} - t_7 t_{10} t_{15} t_{26}, t_2 t_{13} t_{24} - t_{12} t_{26}, t_2 t_{13} t_{23} - t_6 t_9 t_{26},$
$t_2 t_{13} t_{22} - t_4 t_{10} t_{26}, t_2 t_{13} t_{21} - t_4 t_9 t_{26}, -t_{14} t_{19} + t_2 t_{13} t_{20}, t_2 t_{13} t_{18} - t_4 t_{10} t_{19}, t_2 t_{13} t_{16} - t_{10} t_{19},$
$-t_6 t_{10} + t_2 t_{12}, -t_7 t_{10} t_{23} + t_2 t_{11} t_{24}, t_2 t_{11} t_{22} - t_5 t_{10} t_{23}, t_2 t_{11} t_{21} - t_5 t_9 t_{23}, -t_7 t_{14} t_{15} + t_2 t_{11} t_{20},$
$-t_6 t_{14} t_{15} + t_2 t_{11} t_{19}, -t_7 t_{10} t_{15} + t_2 t_{11} t_{16}, t_2 t_{11} t_{13} - t_6 t_9 t_{14}, -t_{10} t_{23} + t_2 t_9 t_{24}, -t_{14} t_{15} + t_2 t_9 t_{20},$
$-t_4 t_{10} t_{15} + t_2 t_9 t_{18}, -t_{10} t_{15} + t_2 t_9 t_{16}, t_2 t_7 t_{21} - t_5 t_{23}, t_2 t_7 t_{19} t_{22} - t_5 t_6 t_{16} t_{26}, -t_5 t_6 t_{16} + t_2 t_7 t_{18},$
$t_2 t_7 t_{13} - t_6 t_{14}, -t_6 t_{22} + t_2 t_4 t_{24}, -t_5 t_{19} + t_2 t_4 t_{20}, -t_5 t_6 t_9 + t_2 t_4 t_{11}, -t_5 t_6 + t_2 t_4 t_7\}$

InvariantBasis[**5**]

```
No more memory available.
Mathematica kernel has shut down.
Try quitting other applications and then retry.
```

The following commands test whether a cycle-monomial is in fact an invariant.

```
T_n_ := DiagonalMatrix[Table[a_i, {i, 1, n}]];
A[n_][Subscript[x, i_, j_]] :=
  (Subscript[T, n].Subscript[X, n].Inverse[Subscript[T, n]])[[i, j]];
MonomialActionList[n_][x_] := Module[{S = {}, F = FactorList[x]}, For[j1 = 2,
    j1 < Length[FactorList[x]] + 1, j1++, S = Join[S, {(A[n][F[[j1, 1]]])^(F[[j1, 2]])}]];
   S];
MonomialAction[n_][x_] := Apply[Times, MonomialActionList[n][x]];
IsInvariant[n_, x_] := FullSimplify[MonomialAction[n][x] / x] === 1;
```

```
TestCycleMonomials[n_] :=
  For[j2 = 1, j2 < n! + n - 1 + 1, j2++, Print[IsInvariant[n, CycleMonomials[n][[j2]]]]];
```

```
IsInvariant[2, x_{1,1} x_{1,2}]
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False
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```
TestCycleMonomials[5]
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