A 2-DIMENSIONAL VERSION OF THE RUBIK'S SQUARE

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Abstract. We define a 2-dimensional version of the Rubik's Cube. Then we compute the group of motions in various cases. Computing the sequence of orders of these groups gives a new sequence in The On-Line Encyclopedia of Integer Sequences. This sequence measures the complexity of the game as the size of the game increases. We end the talk with open questions.

Contents

1. Introduction

We all have heard of the Rubik's Cube and always thought of it as a puzzle or game. But did any one of us ever think of it as mathematical concept involving associativity, binary operations, inverses, identities, permutations, order, isomorphic transformations and other Group Theory properties?! Here I will talk about the permutations of the faces of NxNxN cubes and how the order of the group will help us in identifying the complexity of the game.

For those of you that do have not heard of the Rubik's Cube, The Rubik's Cube is a 3-D combination puzzle invented in 1974 by Hungarian sculptor and professor of architecture Ernô Rubik. In a classic Rubik's Cube, each of the six faces is covered by nine stickers, each

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of one of six solid colors (traditionally white, red, blue, orange, green, and yellow, where white is opposite yellow, blue is opposite green, and orange is opposite red, and the red, white and blue are arranged in that order in a clockwise arrangement). How does the Rubik's Cube relate to group theory? Well, we will see that next.

2. Group Theory

The Rubik's cube is a game based on group theory. In this research, we will prove that the Rubik's cube is actually a group and has the four conditions necessary in order for it to be classified as a group. Knowing that the Rubik's Cube is a group will help us find even more attributes of groups that will tell us more about the game and will help us identify the complexity and possibly even the algorithmic solution of the game.

Well, before we can prove that the Rubik's Cube is a group, we need to know the definition of a group. So what is a group?

Definition 2.1. Suppose that:
i) G is a set and * is a binary operation on G,
ii) * is associative,
iii) there is an element e in G such that x*e=e*x=x for all x in G, and
iv) for each element $x \in G$ there is an element $y \in G$ such that $x * y = y * x = e$,
Then G, together with binary operation $*$, is called a group.

This is the formal definition of a group in the Abstract Algebra book by Dan Saracino. However this definition makes sense with written mathematics but in the context of the Rubik's cube, what does the $*$ symbol mean or how can the Rubik's cube be associative or have an identity or inverse element. What would be an element in the Rubik's cube? We will define these later in the paper.

How do we prove that the Rubik's Cube is a Group? First we have to prove that the Rubik's cube set of motions form a binary operation. If the moves are a binary operation, then we are one step closer to proving that the Rubik's Cube is a group. However what is a Binary Operation? Here is the definition.

Definition 2.2. If S is a set, then a binary operation $*$ on S is a function that associates to each ordered pair (s_1,s_2) of elements of S an element of S, which we denote by $s_1 * s_2$

In the context, we ask ourselves, what constitute a operation * on the group? Well such operation * can be defined as a "motion". A "motion" is any finite sequences of 180° rotations of rows or columns.

How do we prove that the Rubik's cube is a set under a binary operation? Well we can try to prove that the operation of two moves produces yet another move.

Proof. Let M_1 and M_2 be motions.Then $M_1 = r_1 \circ r_2 \circ \cdots \circ r_n$ and $M_2 = s_1 \circ s_2 \circ \cdots \circ s_n$ where r_n and s_n are 180° rotations of a column or a row and "$\circ$" means to perform the rotation from left to right. Then $M_1 \circ M_2 = r_1 \circ r_2 \circ \cdots \circ r_n \circ s_1 \circ s_2 \circ \cdots \circ s_n$ is a finite sequence of rotations of columns or rows thus the operation is closed. Therefore by definition $M_1 \circ M_2$ is a motion. □

In other words, if we let M_1 be one move and M_2 be another move, then $M_1 \circ M_2$ is a move where you let M_1 be the first move followed by M_2. Since the composition of two moves produces yet another moves, the set of motions in a Rubik's Cube forms a binary operation. Now that we have proved it is a binary operation, we will try to prove the rest of that conditions that will attribute the Rubik's Set as a Group. The second step now is trying to prove that the Rubik's Cube has an Identity element in it. By this I mean, what possible element e compares to the Rubik's cube such that when this move is performed on the Rubik's cube, the move will make the cube remain as it was. The identity of the Rubik's cube is the empty move e. Any position plus the empty move will remain in the same original position as it was before. But let's take a look at the formal definition

Definition 2.3. The Identity is an element e in G such that $x * e = e * x = x$ for all $x \in G$.

Finding the inverse of the Rubik's cube game is very similar to the identity element. We can think of the inverse of the Rubik's cube in this context ; You have just performed a move, What other move could you perform so that it appears as if you had not moved anything in the first place? Well, for every move M that you do, you can undo that move which will get you back to the beginning and look as if you didn't do anything. Reversing a move would be the inverse of every move you do. This satisfies the original definition of an inverse which is

Definition 2.4. For each element $x \in G$ there is an element $y \in G$ such that $x * y = y * x = e$ and y is known as the Inverse of x.

Proving that the set of motion in a Rubik's cube is associative is actually a bit harder to do. We will prove its associativity by a general proof of observation.

Proof. Every motion is a permutation by the observation and every permutation is a function. Combining motions is the same as composing the associated function. But compositions of functions are associative thus this is associative. □

Now that we have proved all this, we know that the Rubik's Cube is a group and thus we are able to find other properties of Groups such as order, size and even find its similar isomorphisms. We proved that the Rubik's cube is a group but we can also deduce that a face of a Rubik's cube will also be a group for all NxNxN Rubik's cubes. Therefore, the NxN of any Rubik's cube will also be a group. It is the faces of NxNxN cubes that we will be studying and finding more information about their group.

3. Orders of Groups

Now that we have established that the Rubik's cube is a Group, we can find other properties that will tell us more about the group. We can find the Order of the Group. So what is an order?

Definition 3.1. The order of a group G, denoted by $|G|$, is the number of elements in G.

Knowing the definition of order, we will also be able to find the generators of each cube through the rotations of columns and rows and the permutations that will gives us the admissible positions of the cube. So now we have to define what a generator is.

Definition 3.2. A group G is called cyclic if there is an element x $\in$ G such that $G = (x^n | n \in Z) : x$ is then called a generator for G. We will denote the generator x, $< x >$.

In order to be able to find the order of the group, or the size, we will need to determine the permutations of the group so that we can find the order. Here is the formal definition of a permutation

Definition 3.3. If X is a nonempty set, then a one-to-one onto mappings $X \to X$ is called a permutation of X. We have seen that the set of all such permutations forms a group (S_x,$\circ$)

Finding the permutations will allows us to find all the admissible and inadmissible positions in a cube, but since we are only looking at just one face of any particular cube, we will be studying the permutations of NxN faces of cubes, also known as the permutations of NxN squares. Find the order and permutations of these faces will be nearly impossible to do by hand. Consequently, we will use the aid of GAP to help us solve the order and permutations with the $N + N$ generators of each NxN face, which would be the generators generated by the 180° motions caused by the rows and columns.

4. Permutations

Let's now examine the face of the cube and what it would look like. The face of a 3x3 Rubik's cube can be seen and numerated as these:

1	2	3
4	5	6
7	8	9

The generators of this face would be any 180° motions caused by the rows and columns. Our goal now would be to find all the possible admissible positions which would be the permutations. Now you may be asking, what constitutes an admissible position. Well in the following chart, the position is theoretically possible and can be obtained through cheating when playing with the cube, but when the generators are the rows and columns with 180° motions, then there is no way that the following permutation or positions would ever occur or be possible. This is what we would call and an inadmissible position.

2	1	3
4	5	6
7	8	9

The reason this is not an acceptable position is because when only doing 180° motions through the rows and columns, there is no way that 1 will ever permute with 2. 1 only permutes with 3 and as a result this position will never occur.

To show you the pattern, I went ahead and drew you the permutations and tables of the first couple of NxN faces of a cube. Here are the charts and its permutations.

2x2 face

1	2
3	4

Permutations are (1,2),(3,4),(1,3),(3,4)
With 4 generators

3x3 face

1	2	3
4	5	6
7	8	9

Permutations are (1,3),(4,6),(7,9),(1,7),(2,8),(3,9)
With 6 generators.

4x4 face

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

Permutations are (1,4)(2,3),(5,8)(6,7),(9,12)(10,11),(13,16)(14,15),(1,13)(5,9),
(2,14)(6,10),(3,15)(7,11),(4,16)(8,12)
With 8 Generators

5x5 face

1	2	3	4	5
6	7	8	9	10
11	12	13	14	15
16	17	18	19	20
21	22	23	24	25

Permutations are (1,21)(6,16),(2,22)(7,17),(3,23)(8,18),(4,24)(9,19),(5,25)(10,20),
(1,5)(2,4),(6,10)(7,9),(11,15)(12,14),(16,20)(17,19),(21,25)(22,24)
With 10 Generators

6x6 face

1	2	3	4	5	6
7	8	9	10	11	12
13	14	15	16	17	18
19	20	21	22	23	24
25	26	27	28	29	30
31	32	33	34	35	36

Permutations are (1,31)(7,25)(13,19),(2,32)(8,26)(14,20),(3,33)(9,27)(15,21), (4,34)(10,28)(16,22),(5,35)(11,29)(17,23),(6,36)(12,30)(18,24),(1,6)(2,5)(3,4), (17,12)(8,11)(9,10),(13,18)(14,7)(15,16), (19,24)(20,23)(21,22),(25,30)(26,29)(27,28), (31,36)(32,35)(33,34)
with 12 Generators

Having found all the permutations to the faces of the first 6 NxN excluding the trivial 1x1 face, I then used GAP to help me find the order of each of the faces. The order, or the number of elements in each faces dictates how complex will be, and the higher the number, the more difficult it will be to solve the puzzle. Computing the order of each NxN face yielded to us the following results

NxN Face	Order of Group
2x2	24
3x3	96
4x4	165888
5x5	663552
6x6	165112971264

After getting the output for the groups, our next step was to try to identify the sequence that this numbers produced. After going to The On-Line Encyclopedia of Integer Sequences, we were unable to identify this sequence. This was the breaking point in our research. With the help of Dr. Lawton, he and I were able to discover a new sequence of numbers never before studied of discovered. Finding the pattern to the sequence can and will be very helpful in estimating how hard or how easy the puzzle will be to solve. This sequence measures the complexity of the game as the size of the game increase.

5. Isomorphisms

Our goal was to try to find the patter to this sequence so that we could predict the *nth* term or the order of the NxN face. Unfortunately, we were not successful in finding a pattern. However, this does not mean that such a pattern does not exist. Because we were not successful with finding a pattern with the numbers, we tried to find a different approach that might give us results. We tried to look at the different isomorphisms of the group. What is an isomorphism?

Definition 5.1. Let G, H be groups and let $\varphi : G \rightarrow H$ be a function. Then φ is called a homomorphism if for every a and b in G we have

$$\varphi(\text{ab}) = \varphi(\text{a})\varphi(\text{b}).$$

Definition 5.2. Let $G \rightarrow H$ be a homomorphism. Then φ is called an isomorphism if it is a one-to-one onto function. Two groups G and H are said to be isomorphic if there exists an isomorphism from G onto G. If G and H are isomorphic, we write $G \cong H$

Finding the isomorphism of a group may not be so easy to find as the order of the group gets higher and higher. I will most definitely find the aid of GAP in finding the isomorphisms. Initially, we were able to find the isomorphic transformations of the 2x2 and 3x3 faces but we failed on the 4x4 face. However, later, Dr. Lawton was able to find the isomorphic transformation of the 4x4 face. Nevertheless, we were still unsuccessful in finding a pattern to the sequence that would aid us in finding the nth term of the order of the NxN face. Some interesting properties of the groups that we found using GAP were that the 2x2 face which has an order 24 was one of the 15 groups of order 24. The 3x3 face which has order 96 is also one of the 231 isomorphic groups of order 96. We were also successful in find the structure description of the groups.

The structure description of the 2x2 face was

$$S_4 \tag{5.1}$$

the structure description of the 3x3 face was

$$C_2 \times C_2 \times S_4 \tag{5.2}$$

and finally, the structure description of the 4x4 face was

$$(((A_4 \times A_4) \times Z_4) \times ((A_4 \times A_4) \times Z_4)) \times Z_2 \tag{5.3}$$

We originally thought that perhaps, maybe we would be able to find a pattern through the structure description, but the pattern was not visible. I am not saying that no patterns exists, I am just saying that it still has not yet been discovered.

6. Conclusion

In conclusion, even though there was a new sequence found, there are still many other questions left unanswered. Other questions are "What is the pattern of the sequence? How can we predict the nth term of sequence? How can we know what the order of the NxN face will be?"

Even if there results to be no pattern to the sequence, could there still be a pattern in the structure description that might give as a clue as to the future orders of the groups? There is so much more that can be studied and discovered in this area of mathematics, but one semester is not enough time to answer all these question. There is still plenty of research left to do and more discoveries to be made.

G_n denote the group of orders for the NxN Rubik's square. Then using GAP we determined :
$G_2 \approx S_4$
$G_3 \approx S_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2$
$G_4 \approx (((A_4 \times A_4) \times \mathbb{Z}_2) \times ((A_4 \times A_4) \times \mathbb{Z}_2)) \times Z_2$

References

[1] David Joyner Adventures in Group Theory Rubik's Cube The John Hopkins University Press 2002.
[2] Dan Saracino Abstract Algebra A First Course Waveland Press Second Edition 2008.
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