Hyperbolic, Dual and Complex Numbers

Ramiro Castillo

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Abstract: We define the hyberbolic, complex, and dual numbers and show that they are the only examples of two dimensional number systems. We then look at Euler's formula in all three systems.

Definition A **commutative ring with unity** is a nonempty set R with two operations (usually written as addition and multiplication) that satisfy the following axioms. For all $a, b, c \in R$:

1. *Closure Addition:* $a \in R$ and $b \in R$, then $a + b \in R$.
2. *Associative Addition:*$(a + b) + c = a + (b + c)$.
3. *Additive Identity:* $0 + a = 0 + a = a$.
4. *Commutative Addition:* $a + b = b + a$.
5. *Additive Inverse:* $a + (-a) = (-a) + a = 0$.
6. *Closure Multiplication:* $a \in R$ and $b \in R$, then $ab \in R$.
7. *Associative Multiplication:* $(ab)c = a(bc)$.
8. *Distributive laws:* $a(b + c) = ab + ac$ and $(a + b)c = ac + bc$.
9. *Commutative Multiplication:* $ab = ba$.
10. *Multiplicative identity:* An element $1 \in R$ with $1 \neq 0$ then $1 * a = a = a * 1$

Some Example's of Commutative Rings

1. $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$, and $\mathbb{C}$ are commutative rings with unity.
2. $\mathbb{Z}[i] = a + bi : a, b \in \mathbb{Z}$ is a commutative ring with unity.
3. The ring of even integers is a commutative ring with no unity
4. The set of all $2x2$ matrices is a noncommutative ring.

Definition An **Ideal** is a sub-ring, I of a ring R is called a (two-sided) ideal of R if for every $r \in R$ and every $a \in I$ both ra and ar are in I. A nonempty subset I of a ring R is an ideal of R if:

1. $a, b \in I$, then $a - b \in I$.
2. $r \in R$ and $a \in I$, then $ra \in I$ and $ar \in I$.

Definition Quotient Ring Let R be a commutative ring with unity and let I be an ideal of R. Since R is a group under addition and I is a normal subgroup of R, we may form the factor group

$$R/I = \{r + I | r \in R\}.$$

The set of cosets $\{r + I | r \in R\}$ is a ring under the operations:

1. $(s + I) + (t + I) = (s + t) + I$ and
2. $(s + I)(t + I) = st + I$

if and only if I is an ideal of R.

Proposition 0.1. *R/I is a commutative ring with identity.*

Proof. Let us show that R/I is a factor group.From Ideal is Additive Normal Subgroup that I is a normal subgroup of R under +.Thus, the quotient group (R/I,+) is defined, and as a Factor Group is a Group, + is well-defined. Since multiplication is a binary operation on the cosets let us check that the multiplication is associative and distributive over addition. So let us show that multiplication is well defined if and only if I is an ideal of R.

From Equal Cosets iff Product with Inverse in Coset, we have:
$x_1 + I = x_2 + I \to x_1 + (x_2) \in I$
$y_1 + I = y_2 + I \to y_1 + (y_2) \in I$

hence from the definition of Ideal:
$(x_1 + (x_2))y_1 \in I$
$x_2(y_1 + (y_2)) \in I$

Thus:
$(x_1 + (x_2))y_1 + x_2(y_1 + (y_2)) \in I$
$\to x_1y_1 + ((x_2y_2)) \in I$
$\to x_1y_1 + I = x_2y_2 + I$
From ring properties we have equal Cosets iff Product with Inverse in Coset. □

Definition Polynomial Ring Let $\mathbb{R}$ be the field with real numbers then there exists a ring $\mathbb{R}$[x] that contains an element x not in $\mathbb{R}$ and has the properties:

1. $\mathbb{R}$ is a subring of $\mathbb{R}$[x].
2. $xr = rx$ for all $r \in \mathbb{R}$, where x lies in the center of $\mathbb{R}$[x].
3. Every nonzero element of $\mathbb{R}$[x] can be uniquely written in the form $a_0 + a_1x + \cdots + a_nx^n$ for some $n \geq 0$ $a_i \in \mathbb{R}$ and $a_n \neq 0$.

$$\mathbb{R}[x] = \{s_nx^n + s_{n-1}x^{n-1} + \cdots + s_1x + s_0, s_i \in \mathbb{R}\}$$

is called the ring of polynomials over $\mathbb{R}$. If $f(x) = s_nx^n+s_{n-1}x^{n-1}+\cdots+s_1x+s_0$ is a polynomial over $\mathbb{R}$, then s_n is called the leading coefficient and n is called the degree of f(x).

Definition Let the dual numbers be the quotient ring $\mathbb{D}$=$\mathbb{R}[x]/\langle x^2\rangle$, where the hyperbolic numbers $\mathbb{H} = \mathbb{R}[x]/\langle x^2 - 1\rangle$, and the complex numbers $\mathbb{C} = \mathbb{R}/\langle x^2 + 1\rangle$.

Theorem 0.2. *Division Algorithm Let F be a field,and let $f, g \in F[x]$. Suppose that $f \neq 0$, then there exists unique integers $q, r \in F[x]$ such that*

g(x) = q(x)f(x) + r(x), and degree r(x) < degree f(x) or $r(x) = 0$.

Definition A **field** is a commutative ring with unity in which every nonzero element is a unit.

Corollary 0.3. *From the book "Contemporary Abstract Algebra", Gallian points out $F[x]/\langle p(x)\rangle$ is a Field. Let F be a field and p(x) an irreducible polynomial over F. Then $F[x]/\langle p(x)\rangle$ is a field.*

Proposition 0.4. *$\mathbb{C}$ is a field.*

Proof. Since $\mathbb{R}[x]$ is a commutative ring with identity, so is $\frac{\mathbb{R}[x]}{\langle x^2+1\rangle}$. Let us show that $\frac{\mathbb{R}[x]}{\langle x^2+1\rangle}$ is a field on $\mathbb{C}$. Let $\mathbb{R}$[x] denote the ring of polynomials with real coefficients and let $\langle x^2 + 1\rangle$ denote the ideal generated by $x^2 + 1$, let $f \in \mathbb{R}[x]$. Then by the division algorithm,

$$f(x) = q(x)(x^2 + 1) + r(x)$$

where degree $r(x) < degree(x^2 + 1) = 2$. Thus, $r(x) = ax + b$, for some $a, b \in \mathbb{R}[x]$. So every element of $\mathbb{R}[x]/\langle x^2 + 1\rangle$ is of the form

$$\begin{aligned} f(x)+ < x^2 + 1 > &= q(x)(x^2 + 1) + r(x) + \langle x^2 + 1\rangle \\ &= r(x) + \langle x^2 + 1\rangle \\ &= ax + b + \langle x^2 + 1\rangle. \end{aligned}$$

Since

$$\begin{aligned} x^2 + 1 + \langle x^2 + 1\rangle &= 0 + \langle x^2 + 1\rangle, \textit{we have} \\ x^2 + \langle x^2 + 1\rangle &= -1 + \langle x^2 + 1\rangle \end{aligned}$$

$ax + b + < x^2 + 1 > = 0 + < x^2 + 1 >$
$ax + b = 0 + f(x)(x^2 + 1)$
where 1 = degree (ax+b) = degree ($f(x^2 + 1)$) ≥ 2. Thus, $a = 0$ and $b = 0$ thus for all ax + b so $a \neq 0$ and $b \neq 0$. So we need to find cx + d, where $(ax + b)(cx + d) = 1 + f(x)(x^2 + 1)$ for some f(x).

$$\begin{aligned} \textit{Indeed, } (ax+b)(cx+d) &= 1 + f(x)(x^2+1) \\ acx^2 + adx + bcx + bd &= 1 + f(x)(x^2+1) \\ ac(-1) + adx + bcx + bd &= 1 + f(x)(x^2+1) \\ (ad+bc)x + (bd-ac) &= 1 + f(x)(x^2+1) \end{aligned}$$

$$ad + bc = 0 \textit{ and } bd - ac = 1 \quad (1)$$

$$-b(ad + bc = 0) \rightarrow -abd - b^2c = 0 \quad (2)$$

$$a(bd - ac = 1) \rightarrow abd - a^2c = a \quad (3)$$

$$\textit{by adding equations } (2) \textit{ and } (3) \textit{ it gives us } -abd + abd - b^2c - a^2c = a$$

$$\textit{So, let us solve for (c) } -b^2c - a^2c = a$$

$$\textit{factoring out a (c) gives us } c(-a^2 - b^2) = a$$

$$c = \frac{-a}{a^2 + b^2}$$

$$\textit{so using } (1) \textit{ we have } ad + bc = 0$$

$$\textit{solving for (d) gives us } d = \frac{-bc}{a} \rightarrow d = \frac{-b(\frac{-a}{a^2+b^2})}{a} \rightarrow d = \frac{b}{a^2 + b^2}$$

$$\text{so, } (ax + b)^{-1} = (cx + d) = (\tfrac{-a}{a^2+b^2}x + \tfrac{b}{a^2+b^2}).$$

Let us now show $(ax + b)(cx + d) = 1$ where, $(cx + d) = (\frac{-ax+b}{a^2+b^2})$

$$\begin{aligned} (ax+b)(\tfrac{-ax+b}{a^2+b^2}) &= 1 \\ \tfrac{-a^2x^2+abx-abx+b^2}{a^2+b^2} &= 1 \\ \tfrac{-a^2x^2+b^2}{a^2+b^2} &= 1 \\ \tfrac{-a^2(-1)+b^2}{a^2+b^2} &= 1 \\ \tfrac{a^2+b^2}{a^2+b^2} &= 1 \end{aligned}$$

Thus, $R[x]/\langle x^2 + 1\rangle$ has a multiplicative inverse of $ax + b + \langle x^2 + 1\rangle$. Therefore, $ax + b + \langle x^2 + 1\rangle$ is invertible, so $\mathbb{R}[x]/\langle x^2 + 1\rangle$ is a field on $\mathbb{C}$. □

Proposition 0.5. *$\mathbb{D}$ is not a field.*

Proof. Let us show $\mathbb{D}$ is not a field. So let us find function's $f(x)g(x) = 0$, where $f(x) \neq 0$, $g(x) \neq 0$ and $f(x), g(x) \in \mathbb{D}$. Since $\mathbb{D} = \mathbb{R}[x]/\langle x^2 \rangle$, where $x \neq 0$ and $x \in \langle x^2 \rangle$. $f(x) = x, g(x) = x$,

$$\begin{aligned} \textit{where } f(x)g(x) &= 0 \\ (x)(x) + \langle x^2 \rangle &= 0 + \langle x^2 \rangle \\ x^2 + \langle x^2 \rangle &= 0 + \langle x^2 \rangle \\ (0) + \langle x^2 \rangle &= 0 + \langle x^2 \rangle \end{aligned}$$

therefore, f(x)g(x) = 0 then f^{-1} does not exist. So since it has an [x] that cant be a unit $\mathbb{D}$ is not a field. □

Proposition 0.6. $\mathbb{H}$ *is not a field.*

Proof. Let us show $\mathbb{H}$ is not a field. So let us find function's f(x), g(x) $\in \mathbb{H}$ where f(x)g(x) = 0 and $f(x) \neq 0$, $g(x) \neq 0$. Since $\mathbb{H} = \mathbb{R}[x]/\langle x^2 - 1 \rangle$, then $(x-1) \neq 0$,$(x+1) \neq 0$ and $(x-1), (x+1) \in \langle x^2 - 1 \rangle$.

$$\begin{aligned} \textit{where } f(x)g(x) &= 0 \\ (x+1)(x-1) + \langle x^2 - 1 \rangle &= 0 + \langle x^2 - 1 \rangle \\ x^2 - x + x - 1 + \langle x^2 - 1 \rangle &= 0 + \langle x^2 - 1 \rangle \\ (1) - 1 + \langle x^2 - 1 \rangle &= 0 + \langle x^2 - 1 \rangle \\ 0 + \langle x^2 - 1 \rangle &= 0 + \langle x^2 - 1 \rangle \end{aligned}$$

Since $(x+1) * (x-1) = 0$, where $(x-1), (x+1) \in \langle x^2 - 1 \rangle$, then f^{-1} does not exist. So $\mathbb{H}$ cant be a unit which is not a domain. Hence, it's not a field. □

Acknowledging $\mathbb{H}$, $\mathbb{D}$, and $\mathbb{C}$ we have the following definition.

Definition A 2 dimensional number system is a ring $\mathbb{R}[x]/\langle x^2 + a \rangle$ for $a \in \mathbb{R}$.

Theorem 0.7. $\forall\, a \in \mathbb{R}$, $\mathbb{R}[x]/\langle x^2 + a \rangle \approx$

- $\mathbb{C}$ *if* $a > 0$,
- $\mathbb{H}$ *if* $a < 0$,
- $\mathbb{D}$ *if* $a = 0$.

Proof. Let us show that $\mathbb{R}[x]/\langle x^2 + a \rangle \approx \mathbb{C}$, where $a > 0$ and $a \in \mathbb{R}$.

$$\begin{aligned} x^2 + a + \langle x^2 + a \rangle &= 0 + \langle x^2 + a \rangle \\ x^2 + \langle x^2 + a \rangle &= -a + \langle x^2 + a \rangle. \end{aligned}$$

Thus, the isomorphism $a \to [a] = a + \langle x^2 + a \rangle : x^2 = -a$.

Therefore, the cosets of $\frac{\mathbb{R}[x]}{\langle x^2 + a \rangle}$ are in the form of $ax + b$, for $a, b \in \mathbb{R}$. So by ring homomorphism f:$\frac{\mathbb{R}[x]}{\langle x^2 + a \rangle} \to \mathbb{C}$, by $f(p(x) + \langle x^2 + a \rangle) = p(Ai)$, where $A = \sqrt{a}$. Then,

$$a + bx + \langle x^2 + a \rangle = a + b(Ai) + \langle x^2 + a \rangle \ \textit{where } x = Ai \in \mathbb{C}.$$

so let us show $a + b(Ai) + \langle x^2 + a\rangle = 0 + \langle x^2 + a\rangle$, which means a=0 and b=0. Considering the cosets, $a + bx + \langle x^2 + a\rangle$, $c + dx + \langle x^2 + a\rangle$ then,

$$\begin{array}{rcl} f((a + bx + \langle x^2 + a\rangle) + (c + dx + \langle x^2 + a\rangle)) & = & f(a + c + (b + d)x + \langle x^2 + a\rangle). \\ & = & a + c + (b + d)Ai. \\ & = & a + b(Ai) + c + d(Ai). \\ & = & f(a + bx + \langle x^2 + a\rangle) + f(c + dx + \langle x^2 + a\rangle). \end{array}$$

So, f is a additive group $\mathbb{R}[x]/\langle x^2 + a\rangle$ onto $\mathbb{C}$.

$$\begin{array}{rcl} f((a' + bx + \langle x^2 + a\rangle)(c + dx + \langle x^2 + a\rangle)) & = & f(a'c + (a'd + bc)x + bdx^2 + \langle x^2 + a\rangle). \\ & = & f(a'c - bd(-a) + (ad + bc)x + \langle x^2 + a\rangle). \\ & = & a'c + bd(a) + (ad + bc)(Ai). \\ & = & a'c + bd(a) + a'd(Ai) + bc(Ai). \\ & = & (a' + b(Ai))(c + d(Ai)). \\ & = & f(a' + bx + \langle x^2 + a\rangle)f(c + dx + \langle x^2 + a\rangle). \end{array}$$

Thus, f is a ring now.

Proposition 0.8. *The kernel of a ring morphism is closed under addition, and invariant under multiplication.*

Suppose u + vi is any complex number, then f maps the coset

$$\begin{array}{c} u + v(\frac{x}{A}) + \langle x^2 + a\rangle \text{ to} \\ (u + v(\frac{Ai)}{A}) = u + vi, \end{array}$$

so we see f is surjective, thus f is onto. So let us show that the $Ker(f) = 0 + \langle x^2 + a\rangle$. Therefore, $0 + \langle x^2 + a\rangle \subset Ker(f)$, by

$$\begin{array}{rcl} x^2 + a & = & 0 \text{ , } \textit{where } x = i\sqrt{a} \\ (i\sqrt{a})^2 + a & = & 0 \\ -a + a & = & 0. \end{array}$$

Then let us show that the $Kef(f) \subset \langle x^2 + a\rangle$, by showing f is one-to-one. So,

$$\begin{array}{c} p \in Ker(f) \text{ iff } \mathrm{f(p)} = 0, \\ \text{iff } \mathrm{p(i}\sqrt{a}) = 0 \\ \text{iff } \mathrm{i}\sqrt{a} \text{ is a root of p} \\ \text{iff } x - i\sqrt{a} \mid p \end{array}$$

with p having real coefficients.

Definition $\bar{z} = \overline{a + ib} = a - ib$.

Proposition 0.9.

1. $\overline{zw} = \bar{z}\bar{w}$
2. $\overline{z+w} = \bar{z} + \bar{w}$
3. $\bar{z} = z$ *iff* $z \in \mathbb{R}$

Lemma 0.10. *$p \in \mathbb{R}[x]$, then $p(z) = 0$ iff $p(\bar{z}) = 0$.*

Proof. Les us show $p = \sum_{n=0}^{N} a_n x^n, a_n \in \mathbb{R}\forall n$. Then,

$$\begin{aligned} 0 = \bar{0} = \overline{p(z)} &= \overline{\sum_{n=0}^{N} a_n z^n} \\ &= \sum_{n=0}^{N} \overline{a_n z^n} \\ &= \sum_{n=0}^{N} a_n \bar{z}^n \\ &= p(\bar{z}) \end{aligned}$$

Hence, $p(\bar{z}) = 0$. □

Thus, by lemma 0.9,

$$x + i\sqrt{a} = \overline{x - i\sqrt{a}}.$$

Therefore, $x^2+a = (x+i\sqrt{a})(x-i\sqrt{a})|p \implies p \in \langle x^2+a\rangle$. Hence, $\mathbb{R}[x]/\langle x^2+a\rangle$ is isomorphic to $\mathbb{C}$. □

Proof. Let us show that $\mathbb{R}[x]/\langle x^2 - a\rangle \approx \mathbb{H}$, where $a > 0$. Let $\mathbb{R}$[x] denote the ring of polynomials with real coefficients and let $\langle x^2 - a\rangle$ denote the ideal and let $f \in \mathbb{R}$. Then,

$$\begin{aligned} x^2 - a + \langle x^2 - a\rangle &= 0 + \langle x^2 - a\rangle \\ x^2 + \langle x^2 - a\rangle &= a + \langle x^2 - a\rangle, \textit{where } x = -Aj, x = Aj \in \mathbb{H} \end{aligned}$$

Therefore, the cosets of $\frac{\mathbb{R}[x]}{\langle x^2 - a\rangle}$ are in the form of $ax + b$, for $a, b \in \mathbb{R}$. So,

$$a + bx + \langle x^2 - a\rangle = a + b(Aj) + \langle x^2 - a\rangle.$$

so let us show $a + b(Aj) + \langle x^2 - a\rangle = 0$, which means a=0 and b=0. Considering the cosets, $a + bx + \langle x^2 - a\rangle$, $c + dx + \langle x^2 - a\rangle$ then,

$$\begin{aligned}
f((a+bx+\langle x^2-a\rangle)+(c+dx+\langle x^2-a\rangle)) &= f(a+c+(b+d)x+\langle x^2-a\rangle). \\
&= a+c+(b+d)(Aj)+\langle x^2-a\rangle. \\
&= a+b(Aj)+c+d(Aj)+\langle x^2-a\rangle. \\
&= f(a+bx+\langle x^2-a\rangle)+f(c+dx+\langle x^2-a\rangle).
\end{aligned}$$

So, f is a additive group $\mathbb{R}[x]/\langle x^2-a\rangle$ onto $\mathbb{H}$.

$$\begin{aligned}
f((a+bx+\langle x^2-a\rangle)(c+dx+\langle x^2-a\rangle)) &= f(ac+(ad+bc)x+bdx^2+\langle x^2-a\rangle). \\
&= ac+bd(a')+(ad+bc)(Aj)+\langle x^2-a\rangle. \\
&= ac+bd(a')+ad(Aj)+bc(Aj)+\langle x^2-a\rangle. \\
&= (a+b(Aj))(c+d(Aj))+\langle x^2-a\rangle. \\
&= f(a+bx+\langle x^2+a\rangle)f(c+dx+\langle x^2-a\rangle).
\end{aligned}$$

suppose u + vj is any hyperbolic number, then f maps the coset

$$u+v(\tfrac{x}{A})+\langle x^2-a\rangle \text{ to}$$
$$(u+v(\tfrac{Aj}{A})=u+vj,$$

so we see f is surjective, thus f is a ring homomorphism. Since

$$Ker(f)=0+\langle x^2-a\rangle.$$

Therefore f isomorphic of $\mathbb{R}[x]/\langle x^2-a\rangle$ into $\mathbb{H}$.

□

Let us see what happens for the dual numbers when a = 0, where $a \in \mathbb{R}$.

Proof. For the quotient ring $\mathbb{D}=\mathbb{R}[x]/\langle x^2\rangle$, when $a=0$ in $\mathbb{R}[x]/\langle x^2+a\rangle$ we get $\mathbb{R}[x]/\langle x^2+0\rangle$, which by definition is equal to $\mathbb{D}$. □

One can show $e^{i\pi}+1=0$. Let's see why. How do similar formulas exist for $\mathbb{H}$ and $\mathbb{D}$?

Power Series

Definition Series Expansion

- $e^x = Exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{(1)!} + \frac{x^2}{(2)!} + \frac{x^3}{(3)!} + \cdots,$

- $\cos(x) = \sum_{n=0}^{\infty} \frac{(-1^n)x^{2n}}{(2n)!} = 1 + \frac{-(x)^2}{2!} + \frac{x^4}{4!} + \cdots,$

- $\cosh(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} = 1 + \frac{(x)^2}{2!} + \frac{(x)^4}{4!} + \cdots,$

- $\sin(x) = \sum_{n=0}^{\infty} \frac{(-1^n)x^{2n+1}}{(2n+1)!} = \frac{x}{1!} + \frac{-(x)^3}{3!} + \frac{(x)^5}{5!} \cdots,$

- $\sinh(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} = \frac{(x)}{1!} + \frac{(x)^3}{3!} + \frac{(x)^5}{5!} \cdots.$

Power Series for complex numbers

$$\begin{aligned}\textbf{Exp(ix)} = \sum_{n=0}^{\infty} \frac{(ix)^n}{n!} &= 1 + \tfrac{(ix)}{1!} + \tfrac{-(x)^2}{2!} + \tfrac{-(ix)^3}{3!} + \tfrac{(x)^4}{4!} + \tfrac{(ix)^5}{5!} \cdots \\ &= (1 + \tfrac{-(x)^2}{2!} + \tfrac{x^4}{4!} + \cdots) + i(\tfrac{x}{1!} + \tfrac{-(x)^3}{3!} + \tfrac{(x)^5}{5!} \cdots) \\ &= \cos(x) + i\sin(x)\end{aligned}$$

Therefore,

$$e^{i\pi} = \cos\pi + i\sin\pi = -1 \Longrightarrow e^{i\pi} + 1 = 0.$$

Power Series of Hyperbolic Numbers

$$\begin{aligned}\textbf{Exp(jx)} = \sum_{n=0}^{\infty} \frac{(jx)^n}{n!} &= 1 + \tfrac{(jx)}{1!} + \tfrac{(x)^2}{2!} + \tfrac{(jx)^3}{3!} + \tfrac{(x)^4}{4!} + \tfrac{(jx)^5}{5!} \cdots \\ &= (1 + \tfrac{(x)^2}{2!} + \tfrac{(x)^4}{4!} + \cdots) + j(\tfrac{(x)}{1!} + \tfrac{(x)^3}{3!} + \tfrac{(x)^5}{5!} \cdots) \\ &= \cosh(x) + j\sinh(x)\end{aligned}$$

Because $\cosh(x) = \frac{e^x + e^{-x}}{2}$ and $\sinh(x) = \frac{e^x - e^{-x}}{2}$,

$$\text{then, } e^{\ln(x)j} = \tfrac{x + x^{-1}}{2} + j(\tfrac{x - x^{-1}}{2}).$$

Power Series of Dual Numbers

$$\begin{aligned}Exp(\delta x) = \sum_{n=0}^{\infty} \frac{(\delta x)^n}{n!} &= 1 + \delta x + 0 \cdots \\ &= 1 + \delta x.\end{aligned}$$

Conclusion: From different aspects of complex, dual, and hyperbolic numbers I came to realize that their each unique in their own way. With my mentor Dr.Lawton, I was able to better understand some concepts better for the reason behind their unique 2-dimensional system. For the future I would like to further my research on how to fully understand on how their unique numbers play a role on other algebraic systems.

References

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