Arithmetic of Free Group Character Varieties

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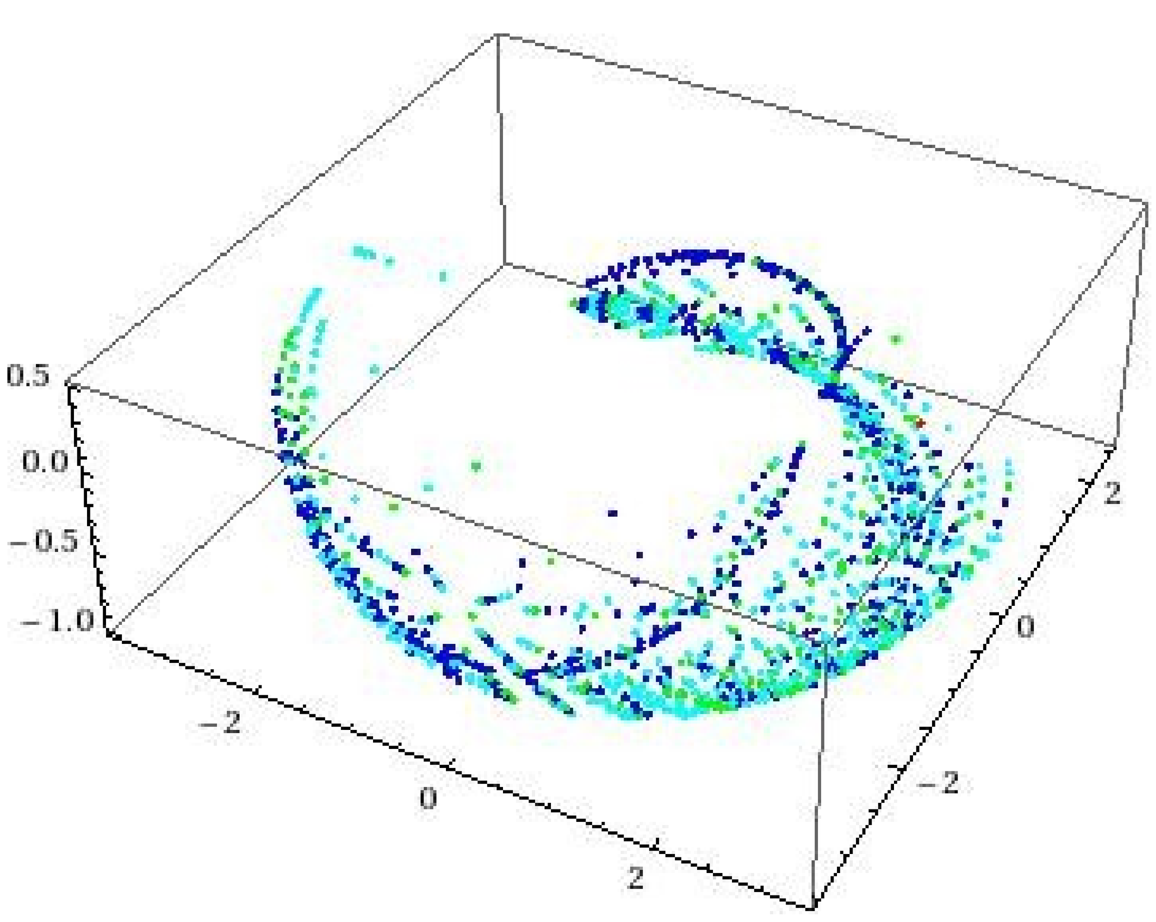
Abstract

A $\mathbb{K}$-variety $X_{\mathbb{K}}$ is a set whose points are the solutions to a set of polynomial equations over a field $\mathbb{K}$. When $\mathbb{K} = \mathbb{Z}_p$, p prime, the counting polynomial $\mathcal{C}_X(p) = |X_{\mathbb{Z}}|$. If $\lim_{p \to 1} \mathcal{C}_X(p) = \chi(X_{\mathbb{C}})$, where χ denotes the Euler Characteristic of the variety X over $\mathbb{C}$, then the variety is said to be of type polynomial count. In 2008, Hausel and Rodriguez-Villegas showed that $Hom(\pi_1(\Sigma)^*, SL(n,\mathbb{C})/SL(n,\mathbb{C})$ is of type polynomial count, where Σ is a closed surface. In 2012 we found that $Hom(\pi_1(\Sigma), SL(2,\mathbb{C})/SL(2,\mathbb{C})$ is not of type polynomial count, where Σ is an open surface.

Terminology.

- **Free Group**: A group with no relations between its generators other than those imposed by the group axioms. The rank of a group is the natural number r which corresponds to the number of generators. For example,

  $F_3 =< x_1, x_2, x_3 >$ denotes a rank 3 free group.
- $\boldsymbol{SL(n, \mathbb{Z}_p)}$: The group of $n \times n$ unit determinant matrices over $\mathbb{Z}_p$.
- $\boldsymbol{\pi_1(\Sigma)^*}$: Here, $\pi_1(\Sigma)$ denotes the fundamental group of the surface Σ, and $\pi_1(\Sigma)^*$ is a central extension of the fundamental group.
- **Representation of a Free Group**: A homomorphism from F_r to $SL(n, \mathbb{Z}_p)$.



$\boldsymbol{Hom(F_2, SL(2,\mathbb{F}))}$, $\mathbb{F}$ **a Finite Field.**

Examples

Let Γ be a surface homeomorphic (can be continuously deformed) to a sphere. Then $\chi(\Gamma) = 2$.

Methods

Using the fact that

$$Hom\big(F_r, SL(2,\mathbb{Z}_p)\big) \approx \mathrm{SL}\big(2,\mathbb{Z}_p\big)^{\times r},$$

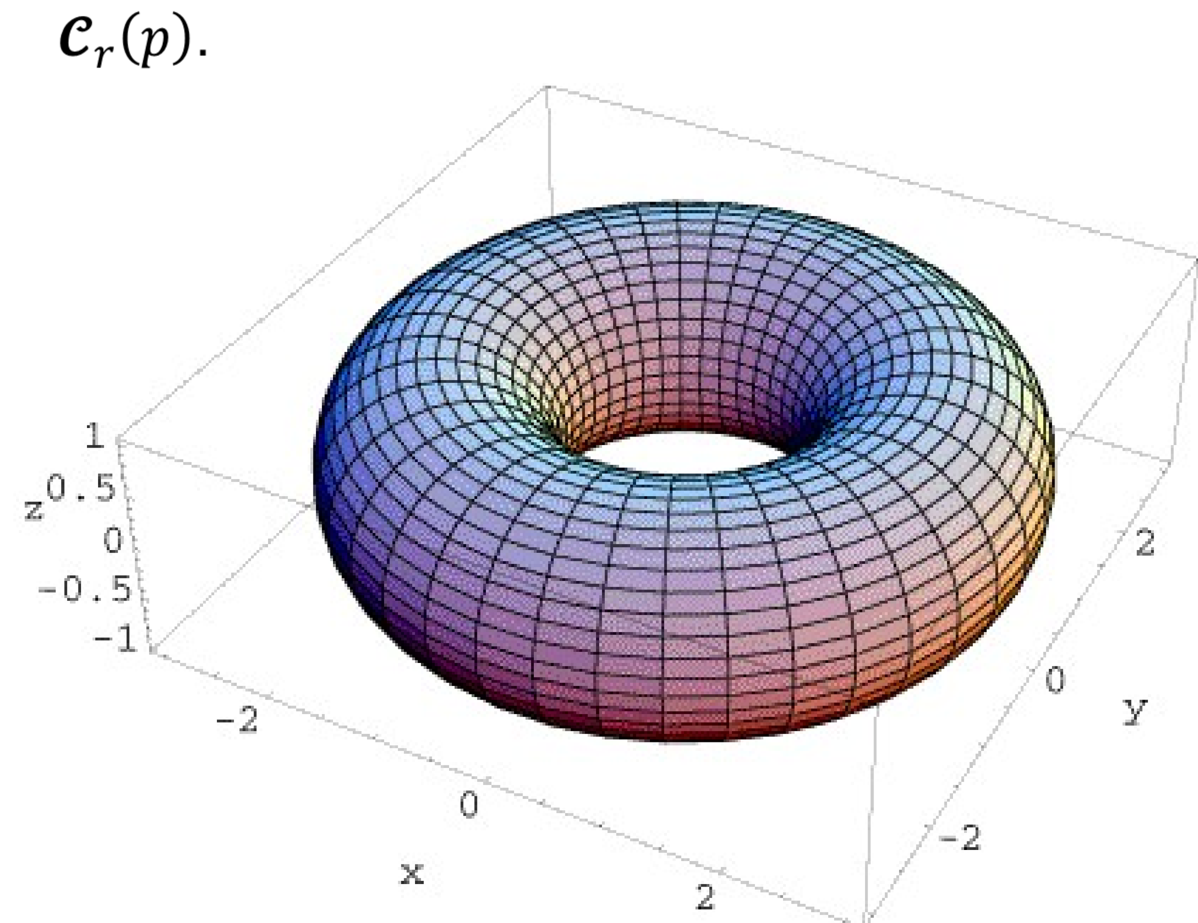
we wrote a program in Mathematica to see $\mathrm{SL}\big(2,\mathbb{Z}_p\big)^{\times r}$ for any given p and r. We then divided the set

$$\mathfrak{X}_{SL(2,\mathbb{Z}_p)}(r) \coloneqq \mathrm{SL}\big(2,\mathbb{Z}_p\big)^{\times r}/SL(2,\mathbb{Z}_p)$$

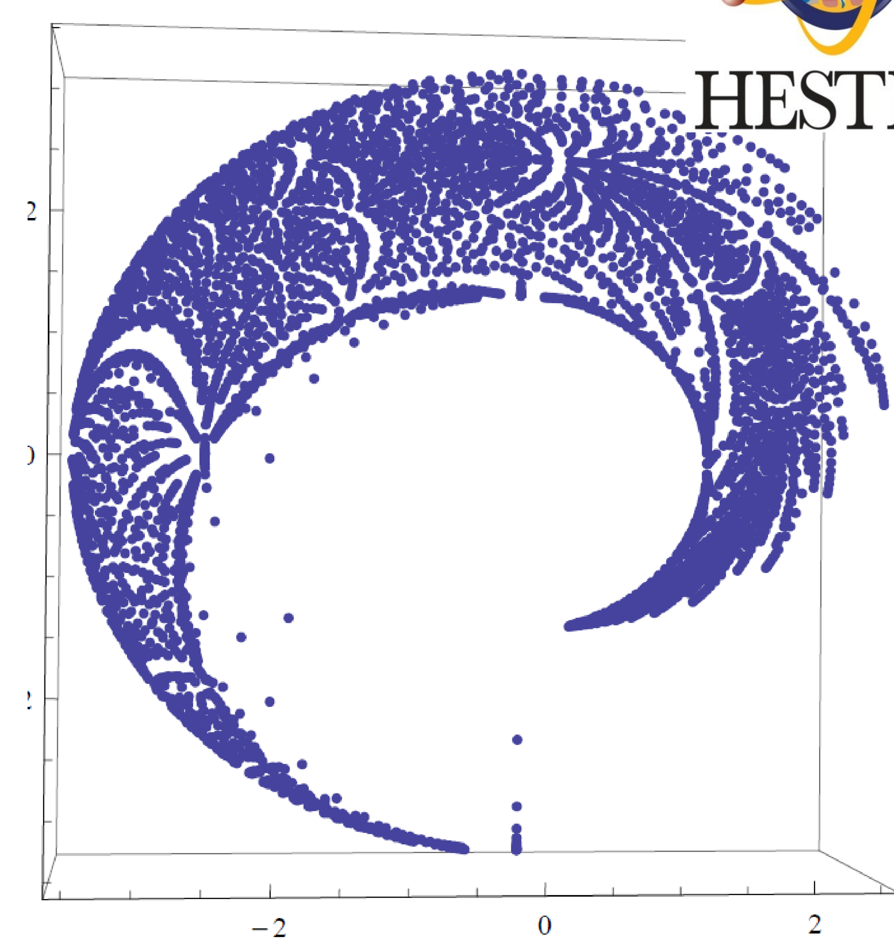
into conjugate invariant strata; i.e. we partitioned $\mathfrak{X}_{SL(2,\mathbb{Z}_p)}(r)$ into sets with fixed stabilizer type. We found that for any given p and r, there are 5 strata (which may be empty) and conjectured a formula to count the number of orbits in each strata in terms of p and r. We then proved that

$$\left|\mathfrak{X}_{SL(2,\mathbb{Z}_p)}(r)\right| = \sum_{i=1}^{5} |\mathcal{R}_I / SL(2, \mathbb{Z}_p)|,$$

Where $\mathcal{R}_i$ denotes the i^{th} strata, and used the counts for each $\mathcal{R}_i$ to find $\mathcal{C}_r(p)$.



$\boldsymbol{Hom(F_2, SL(2,\mathbb{F}))}$, $\mathbb{F}$ **an Infinite Field.**



As the size of the field increases, the number of points increases as well.

Results

We found that the counting polynomial $\mathcal{C}_r(p)$ for $\mathfrak{X}_{SL(2,\mathbb{Z}_p)}(r)$ for $r \geq 2$ is given by

$$\mathcal{C}_r(p) = 2p^{r-1}[(p^2-1)+2^r] + \frac{1}{2}(p-1)[p-1)^{r-2}(p-3) + (p+1)^{r+1}].$$

As a result, we found that

$$\lim_{p \to 1} \mathcal{C}_r(p) = 8 \cdot \chi(\mathfrak{X}_{SL(2,\mathbb{Z}_p)}(r)),$$

Which we found by comparing our result to the known result of $\chi\left(\mathfrak{X}_{SL(2,\mathbb{Z}_p)}(r)\right)^1$.

Hence, since

$$\lim_{p \to 1} \mathcal{C}_X(p) \neq \chi(X_{\mathbb{C}}),$$

we conclude that

$$Hom(\pi_1(\Sigma), SL(2,\mathbb{C})/SL(2,\mathbb{C})$$

is not of type polynomial count.

References

1. *The Topology of moduli spaces of free group representations*, Sean Lawton and Carlos Florentino, 2009, Mathematische Annalen vol. 345.
2. *Mixed Hodge Polynomials of Character Varieties*, Tamàs Hausel and Fernando Rodriguez-Villegas, 2008, Inventiones Mathematicae vol. 174.